Chapter 1

Introduction and basic concepts

1.1 Dynamical systems

Definition 1.1. A *dynamical system* is a triple $(X, G, \mathcal{S})$ defined as the action $\mathcal{S}$ of a semigroup G with identity e on a set X, that is a function

$$\mathcal{S} : G \times X \to X$$

such that $\mathcal{S}(e, x) = x$ for all $x \in X$, and $\mathcal{S}(g_1, \mathcal{S}(g_2, x)) = \mathcal{S}(g_1 g_2, x)$ for all $g_1, g_2 \in G$ and all $x \in X$.

In the following, the set X is assumed to be a locally compact connected metric space.

Two main examples of dynamical system are given in the following definitions.

Definition 1.2. A *discrete-time dynamical system* is defined by the action of $\mathbb{N}_0$ on a set X defined through the iterations of a map $T : X \to X$ by

$$\mathcal{S}(n, x) = T^n(x),$$

where $T^n = T \circ \cdots \circ T$ is the composition of T with itself n times. A discrete-time dynamical system is denoted by the triple $(X, \mathbb{N}_0, T)$.

If the map T is invertible, the system can be extended to the action of the group $\mathbb{Z}$ on X. Examples of a discrete-time dynamical system are sequences defined by a recurrence relation. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_0 = a \in \mathbb{R}, \qquad x_n = f(x_{n-1}) \quad \forall\, n \geq 1,$$

for a real-valued function f. This corresponds to the dynamical system defined on $X = \mathbb{R}$ through the iterations of the map $f : \mathbb{R} \to \mathbb{R}$, that is $x_n = f^n(a)$.

Definition 1.3. A *continuous-time dynamical system* is defined by the action of $\mathbb{R}$ on a set $X \subset \mathbb{R}^n$ defined through the flow $\phi_t(\underline{x})$ of an autonomous ordinary differential equation $\dot{\underline{x}}(t) = F(\underline{x})$, that is

$$\mathcal{S}(t, \underline{x}) = \phi_t(\underline{x}),$$

where $\phi_t(\underline{x})$ is the solution of an ordinary differential equation[1] with initial condition $\underline{x}$, and $\phi_t : X \to X$ is a continuous function. A continuous-time dynamical system is denoted by the triple $(X, \mathbb{R}, \phi)$.

Definiton 1.3 includes the case of non-autonomous differential equations by using the standard procedure of "enlarging" the space of variables. Let $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ define a time-dependent vector field $F(t, \underline{x})$ on $\mathbb{R}^n$ and consider the Cauchy problem

$$\begin{cases} \dot{\underline{x}}(t) = F(t, \underline{x}(t)) \\ \underline{x}(0) = \underline{x}_0 \end{cases}$$

If we let $\underline{y} = (\underline{x}, t) \in \mathbb{R}^{n+1}$ and $\tilde{F}(\underline{y}) = (F(t, \underline{x}), 1)$ be a vector field on $\mathbb{R}^{n+1}$, the previous non-autonomous Cauchy problem is equivalent to the autonomous problem

$$\begin{cases} \dot{\underline{y}}(t) = \tilde{F}(\underline{y}(t)) \\ \underline{y}(0) = (\underline{x}_0, 0) \end{cases}$$

A similar procedure can be applied to the case of sequences defined by a recurrence relation depending on n.

Analogously, it is known that ordinary differential equations of order greater than one can be reduced to systems of ordinary differential equations of order one, hence again included in Definition 1.3. The same is true for the discrete-time case. The following example shows how the procedure works.

[1]All ordinary differential equations we consider are assumed to have the property of local uniqueness of solutions and time-interval of existence of solutions given by $\mathbb{R}$ up to reparametrization.

Example 1.1. Let us consider the sequence $\{x_n\}$ defined as follows

$$x_1 = 0, \quad x_2 = 1, \quad x_3 = 1, \quad x_n = x_{n-1} + 2^{n-3}\, x_{n-2} + x_{n-3} \quad \forall\, n \geq 4.$$

We define the vector $\underline{y}_n = (x_n, x_{n-1}, x_{n-2}, n) \in \mathbb{R}^4$. Then using the previous recurrence we have

$$\underline{y}_{n+1} = \Big(x_n + 2^{n-2}\, x_{n-1} + x_{n-2}\, ,\, x_n\, ,\, x_{n-1}\, ,\, n+1\Big) = T(\underline{y}_n) \quad \forall\, n \geq 3$$

with initial condition set to be $y_3 = (1, 1, 0, 3)$ and $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by

$$T(a, b, c, d) = \Big(a + 2^{d-2}\, b + c\, ,\, a\, ,\, b\, ,\, d+1\Big).$$

The idea of an action of a semigroup on a set X can be used in more abstract contexts. Here we show only one example of algebraic nature that will be studied in more details in part IV of this book.

Example 1.2. Let X be a group, G be $\mathbb{R}$, and consider the action S on X given by multiplication for a one-parameter subgroup of X. For example, if $X = SL(2, \mathbb{R})$ the action of $\mathbb{R}$ defined by

$$S(t, x) = x \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2, \mathbb{R})$$

represents the geodesic flow on the hyperbolic Poincaré half-plane (see Chapter 9).

1.2 Basic notions

Definition 1.4. Given a dynamical system $(X, G, \mathcal{S})$, the *orbit* of a point $x \in X$ is the set $\mathcal{O}(x) := \{\mathcal{S}(g, x)\, :\, g \in G\}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the orbit of a point $x \in X$ is the set

$$\mathcal{O}(x) = \{T^n(x)\, :\, n \in \mathbb{N}_0\}\,. \tag{1.1}$$

If the map T is invertible, then we can consider the action of the group $\mathbb{Z}$ on X and define the *forward orbit* and *backward orbit* of a point $x \in X$ by

$$\mathcal{O}^+(x) := \{T^n(x)\, :\, n \geq 0\}\,, \quad \mathcal{O}^-(x) := \{T^n(x)\, :\, n \leq 0\}\,.$$

The orbit $\mathcal{O}(x)$ is then given by $\mathcal{O}^+(x) \cup \mathcal{O}^-(x)$.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *forward orbit* and *backward orbit* of a point $\underline{x} \in X$ are defined by

$$\mathcal{O}^+(\underline{x}) := \bigcup_{t \geq 0} \phi_t(\underline{x}), \qquad \mathcal{O}^-(\underline{x}) := \bigcup_{t \leq 0} \phi_t(\underline{x}), \tag{1.2}$$

and the orbit is $\mathcal{O}(\underline{x}) = \mathcal{O}^+(\underline{x}) \cup \mathcal{O}^-(\underline{x})$.

Definition 1.5. Given a dynamical system $(X, G, \mathcal{S})$, the *centralizer* of a point $x \in X$ is the sub-semigroup

$$\mathcal{C}(x) := \{g \in G \,:\, S(g, x) = x\}\,.$$

A point x is called *fixed* if $\mathcal{C}(x) = G$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, a point $x \in X$ is fixed if and only if $T(x) = x$. If x is not a fixed point but its centralizer is not G, x is called *periodic* and the minimum positive element in $\mathcal{C}(x)$ is the *minimal period* of x. For a fixed point $\mathcal{O}(x) = \{x\}$, and for a periodic point of minimal period p

$$\mathcal{O}(x) = \left\{x,\, T(x),\, T^2(x),\, \ldots,\, T^{p-1}(x)\right\}.$$

For a non-invertible map there might be points which are not periodic but are pre-images of a periodic point. For such points x, the centralizer contains only the identity of G, but there exists $k \geq 1$ such that $\mathcal{C}(T^k(x))$ has a minimal positive element p. These points are called *pre-periodic with minimal period* p.

For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$ given by the solutions to $\dot{\underline{x}}(t) = F(\underline{x})$, a point $\underline{x} \in X$ is fixed if and only if $F(\underline{x}) = \underline{0}$. If $\underline{x}$ is not a fixed point but its centralizer is not trivial, $\underline{x}$ is called *periodic* and the minimum positive element in $\mathcal{C}(\underline{x})$ is the *minimal period* of T. A periodic point $\underline{x}$ of minimal period $T > 0$ satisfies

$$\phi_{t+T}(\underline{x}) = \phi_t(\underline{x}), \quad \forall\, t \in \mathbb{R},$$

and

$$\phi_{t+s}(\underline{x}) \neq \phi_t(\underline{x}), \quad \forall\, s \in (0, T),\, t \in \mathbb{R}.$$

For a fixed point $\mathcal{O}(\underline{x}) = \{\underline{x}\}$. For a periodic point of minimal period T

$$\mathcal{O}(\underline{x}) = \bigcup_{0 \leq t \leq T} \phi_t(\underline{x}),$$

and its orbits is called *a periodic orbit of period* T.

Definition 1.6. Given a dynamical system $(X, G, \mathcal{S})$, a set $A \subset X$ is called *invariant* if for each $x \in A$ it holds $\mathcal{S}(g, x) \in A$ for all $g \in G$.

For a continuous-time dynamical system one can introduce a weaker notion. We say that a subset A of X is *forward invariant* if for each $\underline{x} \in A$ it holds $\phi_t(\underline{x}) \in A$ for all $t \geq 0$. Analogously A is called *backward invariant* if the same relation holds for all $t \leq 0$. By definition, A is *invariant* if the previous relation holds for all $t \in \mathbb{R}$.

For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, we consider more situations. We say that a subset A of X is *forward invariant* if $T(A) \subseteq A$, A is called *fully invariant* if $T(A) = A$, A is called *completely invariant* if $T^{-1}(A) = A$. The different notions are useful in different approaches.

Finally, if the action of the group G on X can be interpreted in terms of time evolution, we can introduce notions about the forward and backward evolution of an orbit. In more general situations, one studies the set of all the possible limit points of an orbit as the sequence of the elements of the group acting varies.

Definition 1.7. For a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, the *ω-limit set* of a point $x \in X$ is the set

$$\omega(x) := \{y \in X \,:\, \exists\, n_k \to +\infty \text{ such that } T^{n_k}(x) \to y \text{ as } k \to \infty\}.$$

Definition 1.8. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the *α-limit set* of a point $\underline{x} \in X$ is the set

$$\alpha(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to -\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Analogously the *ω-limit set* of a point $\underline{x} \in X$ is the set

$$\omega(\underline{x}) := \left\{\underline{y} \in X \,:\, \exists\, t_k \to +\infty \text{ such that } \phi_{t_k}(\underline{x}) \to \underline{y} \text{ as } k \to \infty\right\}.$$

Proposition 1.1. *Given a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, let $\underline{x} \in X$ such that $\mathcal{O}^+(\underline{x})$ is bounded. Then the set $\omega(\underline{x})$ is non-empty, compact and invariant. If $\mathcal{O}^-(\underline{x})$ is bounded, the same holds for the set $\alpha(\underline{x})$.*

Proof (see [Gl94]). Given a point $\underline{x}$ with bounded forward orbit, let us consider a strictly increasing sequence $\{\tau_j\}_{j=0}^{\infty}$ of times in $\mathbb{R}^+$ with $\tau_0 = 0$ and $\tau_j \to +\infty$, and let $\underline{x}_j := \phi_{\tau_j}(\underline{x})$. We first show that

$$\omega(\underline{x}) = \bigcap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}. \tag{1.3}$$

By the definition of the ω-limit set, it is immediate that $\omega(\underline{x}) \subset \overline{\mathcal{O}^+(\underline{x}_j)}$ for all $j \geq 0$. Hence it remains to show that if $\underline{y} \in \cap_{j=0}^{\infty} \overline{\mathcal{O}^+(\underline{x}_j)}$ then $\underline{y} \in \omega(\underline{x})$. By definition of closure of a set, for all $j \geq 0$ there exists a sequence $\{\underline{\xi}_n^j\}_n$ of points in $\mathcal{O}^+(\underline{x}_j)$ such that $\underline{\xi}_n^j \to \underline{y}$, hence there exists a sequence $\{t_n^j\}_n$ such that $\phi_{t_n^j}(\underline{x}_j) \to \underline{y}$. In particular we have proved that there exists a strictly increasing diverging sequence $\{\tau_j\}_{j=0}^{\infty}$ and sequences $\{t_n^j\}_n$ such that

$$\phi_{\tau_j + t_n^j}(\underline{x}) \xrightarrow[n\to\infty]{} \underline{y}, \quad \forall\, j \geq 0.$$

From $\{\tau_j + t_n^j\}_{j,n}$ we can then extract a diverging sequence $\{\tilde{t}_k\}_k$ such that $\phi_{\tilde{t}_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. Hence $\underline{y} \in \omega(\underline{x})$, and (1.3) is proved.

The first properties of $\omega(\underline{x})$ follow from (1.3). The sets $\{\overline{\mathcal{O}^+(\underline{x}_j)}\}_j$ define a decreasing sequence of non-empty closed sets, which are bounded because $\mathcal{O}^+(\underline{x})$ is bounded. Hence $\omega(\underline{x})$ is a non-empty compact set. It remains to prove that it is invariant.

Let $\underline{y} \in \omega(\underline{x})$, and let $\{t_k\}_k$ be a positively diverging sequence such that $\phi_{t_k}(\underline{x}) \to \underline{y}$ as $k \to \infty$. By the properties of a continuous-time dynamical system

$$\phi_{t+t_k}(\underline{x}) = \phi_t(\phi_{t_k}(\underline{x})) \xrightarrow[k\to\infty]{} \phi_t(\underline{y}), \quad \forall\, t \in \mathbb{R}.$$

Hence we have shown that $\phi_t(\underline{y}) \in \omega(\underline{x})$ for all $t \in \mathbb{R}$. This concludes the proof for the ω-limit set.

The proof for the α-limit set follows along the same lines. □

Proposition 1.2. *Given a discrete-time dynamical system $(X, \mathbb{N}_0, T)$, let $x \in X$ such that $\mathcal{O}(x)$ is bounded. Then the set $\omega(x)$ is non-empty and compact. If T is continuous then $\omega(x)$ is fully invariant.*

Proof. We can repeat the proof of Proposition 1.1 to show that the ω-limit set is non-empty and compact. In particular the proof follows from the analogue of (1.3).

Let $T : X \to X$ be a continuous map with respect to a topological structure on X. Then given $y \in \omega(x)$, and being $\{n_k\}_k$ the diverging sequence of naturals for which $T^{n_k} \to y$ as $k \to \infty$, we have

$$T^{n_k+1}(x) = T(T^{n_k}(x)) \xrightarrow[k\to\infty]{} T(y).$$

Hence $T(y) \in \omega(x)$, and $\omega(x)$ is a positively invariant set. On the other hand, since $\mathcal{O}(x)$ is bounded, the sequence $\{T^{n_k-1}(x)\}_k$ admits a convergent sub-sequence $\{T^{n_{k_j}-1}(x)\}_j$ with limit point z. Hence $z \in \omega(x)$. Again by continuity of T we find

$$T(z) = T\Big(\lim_{j\to\infty} T^{n_{k_j}-1}(x)\Big) = \lim_{j\to\infty} T^{n_{k_j}}(x) = y$$

since n_{k_j} is a subsequence of n_k. Hence $y \in T(\omega(x))$, and $\omega(x)$ is then fully invariant. □

We remark that the ω-limit set is not completely invariant in general. It is sufficient to think of the case in which the ω-limit set is a fixed point with more than one pre-image.

Definition 1.9. For a continuous-time dynamical system $(X, \mathbb{R}, \phi)$, the orbit of a point $\underline{y}$ is called *homoclinic* if there exists a fixed point $\underline{x}$ such that

$$\alpha(\underline{y}) = \omega(\underline{y}) = \{\underline{x}\}.$$

If there exist two distinct fixed points $\underline{x}_1$, $\underline{x}_2$ such that

$$\alpha(\underline{y}) = \{\underline{x}_1\} \quad \text{and} \quad \omega(\underline{y}) = \{\underline{x}_2\},$$

then the orbit of the point $\underline{y}$ is called *heteroclinic.*

Definition 1.9 can be adapted verbatim to the case of a discrete-time dynamical system $(X, \mathbb{N}_0, T)$ with invertible T.

1.3 Examples

Here we collect the main examples of discrete dynamical systems that will be used in the following.

Example 1.3 (The roots). Sequences defined by a recurrence are the first very basic example of a discrete-time dynamical system. Let $c > 0$, $k \in [1, 3]$, and consider the sequence $\{a_n\}$ defined by

$$\begin{cases} a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n^k}\right), & \forall\, n \geq 0 \\ a_0 \in (0, +\infty) \end{cases}$$

It is an exercise to prove that for all $a_0 \in \mathbb{R}^+$ it holds $\lim_n a_n = c^{\frac{1}{k+1}}$. This can be read as a result about the asymptotic behaviour of the orbits of points in $\mathbb{R}^+$ for the dynamical system defined by the map

$$T_{c,k} : \mathbb{R}^+ \to \mathbb{R}^+, \quad T_{c,k}(x) = \frac{1}{2}\left(x + \frac{c}{x^k}\right).$$

In fact one can prove that $\omega(x) = c^{\frac{1}{k+1}}$ for all $x \in \mathbb{R}^+$.

Example 1.4 (Rotations of the circle). Let us consider the action of $\mathbb{Z}$ on S^1 given by the rotation of an angle $2\pi\alpha$, for $\alpha \in \mathbb{R}$, that is

$$\mathcal{S}(n, z) = z\, e^{2\pi i n\alpha} \in S^1, \quad \forall\, z \in S^1,\, n \in \mathbb{Z}.$$

By writing $S^1 = \{z \in \mathbb{C} \,:\, z = e^{2\pi i x}, x \in \mathbb{R}\}$, we make the identification of S^1 with $[0,1]/(0 \sim 1)$, the unit interval with end points identified. The rotation of angle $2\pi\alpha$ can then be written as a map on S^1 as

$$R_\alpha : S^1 \to S^1, \quad R_\alpha(x) = x + \alpha \pmod 1 = \{x + \alpha\}. \tag{1.4}$$

Proposition 1.3. *If α is rational all orbits of R_α are periodic of the same minimal period. If α is irrational all orbits of R_α are dense.*

Proof. If $\alpha = p/q \in \mathbb{Q}$ with $(p,q) = 1$, then $R^q_\alpha(x) = \{x + q\alpha\} = x$ for all $x \in [0,1)$. In addition, if $n \in \mathbb{N}$ and $n < q$, we can write $n\alpha = np/q = m + r/q$ with $m \in \mathbb{Z}$ and $r/q \in \mathbb{Q} \cap (0,1)$. Hence $R^n_\alpha(x) = \{x + r/q\} \neq x$. It follows that all orbits are periodic of the minimal period q.

Let's now assume that α is irrational. Since R_α is an isometry, it is enough to show that one orbit is dense. In fact, we prove that forward orbits are dense by considering $\{R^n_\alpha(0)\}_{n \geq 0}$.

Let $x \in S^1$, then we show that for any $\varepsilon > 0$ there exists $\bar{n}$ such that $R^{\bar{n}}_\alpha(0) \in (x - \varepsilon\,,\, x + \varepsilon)$. First, by Proposition B.3, we find $p, q \in \mathbb{N}$ such that $0 < q\alpha - p < \varepsilon$. This means that $R^q_\alpha(0) \in (-\varepsilon, \varepsilon)$. If we now consider the points $\{k(q\alpha - p)\}_{k \geq 0}$, it follows that there exists $K > 0$ such that the points $\{k(q\alpha - p)\}_{0 \leq k \leq K}$ create a partition of $[0,1]$ into intervals of length less than ε. Therefore for all $x \in S^1$

$$\min_{0 \leq k \leq K} d(x, k(q\alpha - p)) < \varepsilon$$

and the minimum is achieved for some value $\bar{k}$. Hence choosing $\bar{n} = \bar{k}q$ the proof is finished. □

A consequence of the proposition is that if α is rational, then all points have their own periodic orbit as α-limit and ω-limit sets. Instead, if α is irrational, then $\alpha(x) = \omega(x) = S^1$ for all x.

Example 1.5 (The tent maps). It is a family of maps

$$T_s : [0,1] \to [0,1] \quad \text{with } s \in (0,2]$$

defined as

$$T_s(x) = \begin{cases} s\,x, & \text{if } x \in [0, \frac{1}{2}]; \\ s\,(1 - x), & \text{if } x \in [\frac{1}{2}, 1]. \end{cases} \tag{1.5}$$

Example 1.6 (The logistic maps). It is a family of maps

$$T_\lambda : [0,1] \to [0,1] \quad \text{with } \lambda \in (0,4]$$

defined as

$$T_\lambda(x) = \lambda\, x\,(1-x). \tag{1.6}$$

Example 1.7 (Linear endomorphisms of the circle). It is a family of maps

$$T_m : S^1 \to S^1 \quad \text{with } m \in \mathbb{N},\, m \geq 2$$

where again we think of S^1 as $[0,1]/(0 \sim 1)$, defined as

$$T_m(x) = \{mx\}. \tag{1.7}$$

Special cases are $m = 2$ which is also called the *Bernoulli map* and is related with the binary expansion of real numbers, and $m = 10$ which is related with the decimal expansion of real numbers.

Example 1.8 (Symbolic dynamics). We now introduce an abstract system. Let $\mathcal{A}$ be a finite or countable alphabet and denote by $N \in \mathbb{N} \cup \{\infty\}$ the number of symbols. Let $\Omega_{\mathcal{A}}$ be the set of all infinite strings with symbols from $\mathcal{A}$, that is

$$\Omega_{\mathcal{A}} = \mathcal{A}^{\mathbb{N}_0} = \{\omega = (\omega_i)_{i\in\mathbb{N}_0} \,:\, \omega_i \in \mathcal{A}\ \forall\, i \in \mathbb{N}_0\}\,.$$

If $N < \infty$, the space X is compact when endowed with the product topology or with the metric

$$d_\theta(\omega, \tilde{\omega}) := \theta^{\min\{i\in\mathbb{N}_0\,:\,\omega_i\neq\tilde{\omega}_i\}}\,, \quad \text{for a fixed } \theta \in (0,1). \tag{1.8}$$

The space $\Omega_{\mathcal{A}}$ is totally disconnected and a basis of the product topology is given by the *cylinders*: for $k \in \mathbb{N}$, $i_1, i_2, \dots, i_k \in \mathbb{N}_0$, and $a_1, a_2, \dots, a_k \in \mathcal{A}$, we define

$$C_{i_1,i_2,\dots,i_k}(a_1, a_2, \dots, a_k) := \left\{\omega \in \Omega_{\mathcal{A}} \,:\, \omega_{i_j} = a_j\ \forall\, j = 1, \dots, k\right\}.$$

In particular, we use the notations $C(a) = C_1(a)$ and

$$C_{i_1,i_2,\dots,i_k}(\omega) = C_{i_1,i_2,\dots,i_k}(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_k})$$

for a fixed $\omega \in \Omega_{\mathcal{A}}$.

On $\Omega_{\mathcal{A}}$ we consider the discrete dynamical system given by the action of the continuous map

$$\sigma : \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}, \quad (\sigma(\omega))_i = \omega_{i+1} \quad \forall\, i \in \mathbb{N}_0.$$

The system $(\Omega_{\mathcal{A}}, \mathbb{N}_0, \sigma)$ is called *full shift on* $\mathcal{A}$.

In some situations it is useful to consider a sub-system of the full shift. A first easy example is given by considering infinite strings which cannot contain a given set of words of finite length. For example, let $M = (m_{ij}) \in M(N \times N, \{0,1\})$, a $N \times N$ matrix with coefficients in the set $\{0,1\}$ and rows and columns indexed by $\mathcal{A}$. We set

$$\Omega_{\mathcal{A},M} := \left\{ \omega \in \mathcal{A}^{\mathbb{N}_0} \; : \; m_{\omega_i\, \omega_{i+1}} = 1 \; \forall\, i \in \mathbb{N}_0 \right\},$$

that is, saying that the transition from $a \in \mathcal{A}$ to $b \in \mathcal{A}$ is allowed iff $m_{ab} = 1$, the set $\Omega_{\mathcal{A},M}$ contains the infinite strings in $\mathcal{A}^{\mathbb{N}_0}$ which contain only allowed transitions. It is immediate to verify that $\Omega_{\mathcal{A},M}$ is forward invariant for the action of σ and it is fully invariant if for each $b \in \mathcal{A}$ there exists $a \in \mathcal{A}$ with $m_{ab} = 1$. Hence we can restrict the action of σ to $\Omega_{\mathcal{A},M}$, and the dynamical system $(\Omega_{\mathcal{A},M}, \mathbb{N}_0, \sigma)$ is called *subshift of finite type on* $\mathcal{A}$.

Finally, by considering bi-infinite strings $\mathcal{A}^{\mathbb{Z}}$, one can consider the action of σ on $\mathcal{A}^{\mathbb{Z}}$ and on $\mathcal{A}^{\mathbb{Z}}_M$. In this case the map σ is invertible and the dynamical systems $(\mathcal{A}^{\mathbb{Z}}, \mathbb{Z}, \sigma)$ and $(\mathcal{A}^{\mathbb{Z}}_M, \mathbb{Z}, \sigma)$ are called double full shift and double subshift of finite type, respectively.

Example 1.9 (Toral automorphisms). Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ be the two dimensional torus. Given a matrix $A \in M(2 \times 2, \mathbb{Z})$ with $\det(A) = \pm 1$, the linear map $\mathbb{R}^2 \ni \underline{x} \mapsto A\,\underline{x}$ may be projected onto a continuous automorphisms of $\mathbb{T}^2$ given by

$$\mathbb{T}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto T_A(x,y) := A \begin{pmatrix} x \\ y \end{pmatrix} \mod \mathbb{Z}^2.$$

The most famous example is the so-called *Arnold's Cat map*, which is the toral automorphism given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Example 1.10 (The standard map). Let us consider an electron with charge e moving horizontally in a cyclotron thanks to the action of a vertical magnetic field of constant modulus B, and subject to a time-dependent voltage drop $V\sin(\omega t)$ across a narrow azimuthal gap. Let E denote the energy of the electron, then the period of rotation is given by $T = 2\pi\,\frac{E}{eBc}$. We measure energy and time (E,t) just before every voltage drop, hence after one circuit we obtain

$$E' = E - e\,V\sin(\omega t)\,, \qquad t' = t + \frac{2\pi}{eBc}\,E'\,.$$

Using the variables $x := \frac{\omega}{2\pi}t$ and $y := \frac{\omega}{eBc}E$, and setting $k := 2\pi\frac{\omega V}{Bc}$, we have defined the map

$$\tilde{T} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}\,, \quad \tilde{T}(x,y) = \Big(x + y - \frac{k}{2\pi}\,\sin(2\pi x)\,,\, y - \frac{k}{2\pi}\,\sin(2\pi x)\Big)\,.$$

Note that $\tilde{T}(x+1,y) = \tilde{T}(x,y) + (1,0)$, hence given the projection $\pi : \mathbb{R}\times\mathbb{R} \to S^1\times\mathbb{R}$ defined as $\pi(x,y) = (x - \lfloor x \rfloor, y)$, it follows that the map

$$T : S^1\times\mathbb{R} \to S^1\times\mathbb{R}\,, \quad T(x,y) = \Big(x + y - \frac{k}{2\pi}\,\sin(2\pi x)\,,\, y - \frac{k}{2\pi}\,\sin(2\pi x)\Big) \tag{1.9}$$

satisfies $\pi\circ\tilde{T} = T\circ\pi$. Hence $\tilde{T}$ is a lift of T. The map T is known as the *(Chirikov) standard map.*

Note also that $T(x,y+1) = T(x,y) + (0,1)$, hence the standard map can be considered as acting on $\mathbb{T}^2$.

Example 1.11 (Birkhoff billiards). Let $\Omega \subset \mathbb{R}^2$ be a strictly convex domain with C^3 boundary[2]. Let us normalize the set to $|\partial\Omega| = 1$ and fix the positive orientation of the boundary.

The *mathematical billiard* is the continuous dynamical system given by the frictionless motion of a pointwise ball inside Ω, with elastic specular reflections at $\partial\Omega$. The phase space is then given by $\Omega\times S^1$, since the velocity of the ball is preserved in modulus.

A convenient simpler description of the system is given by the Poincaré map of the flow on the set $\partial\Omega\times[0,\pi]$, described by the evolution of the couples (position,angle) of the subsequent collisions of the ball with the boundary of the set. For each collision, its position can be described by the arc-length coordinate $s \in S^1$ and its angle by the angle $\vartheta \in [0,\pi]$ between the trajectory of the ball after the collision and the oriented tangent vector to $\partial\Omega$ at the collision point. We have thus described a map

$$T : S^1\times(0,\pi) \to S^1\times(0,\pi)$$

[2] Thanks to [Ha77] this assumption avoid accumulation of collision times

which can be continuously extended to $S^1 \times [0,\pi]$ by $T(s,0) = (s,0)$ and $T(s,\pi) = (s,\pi)$.

This map may be defined with some cautions for more general domains $\Omega \subset \mathbb{R}^2$ (see [CM06]).

Example 1.12 (Mechanics and Billiards). Let m_1 and m_2 be two distinct point masses moving frictionless on the interval $[0,1]$, subject to perfectly elastic collisions among them and with two infinite ideal walls at the extremes of the interval. Let $x_1, x_2 \in [0,1]$ with $x_1 \leq x_2$, and $v_1, v_2 \in \mathbb{R}$, denote the positions and velocities of the masses, and introduce the variables $q_1 := \sqrt{m_1}\, x_1$ and $q_2 := \sqrt{m_2}\, x_2$, and $u_1 = \sqrt{m_1}\, v_1$ and $u_2 := \sqrt{m_2}\, v_2$. The invariances of the kinetic energy K and of the linear momentum P of the system read in the new variables as

$$u_1^2 + u_2^2 = 2K\,, \quad \sqrt{m_1}\, u_1 + \sqrt{m_2}\, u_2 = P\,.$$

In the new variables, the configuration space is given by the triangle

$$A = \left\{ (q_1, q_2) \in \mathbb{R}^2 \,:\, q_1 \geq 0\,,\, q_2 \leq \sqrt{m_2}\,,\, \sqrt{m_2}\, q_1 \leq \sqrt{m_1}\, q_2 \right\}\,.$$

A trajectory $(q_1(t), q_2(t))$ satisfies the following constraints:

$$\dot{q}_1^2(t) + \dot{q}_2^2(t) = 2K \qquad \forall\, t$$

(hence the motion occurs with constant speed);

$$\sqrt{m_1}\, \dot{q}_1 + \sqrt{m_2}\, \dot{q}_2 = P \qquad \forall\, t$$

(hence the velocity vector of the motion has fixed scalar product with the vector $(\sqrt{m_1}, \sqrt{m_2})$).

These properties imply that the motion $(q_1(t), q_2(t))$ in A can be described by the orbit of a mathematical billiard ball inside A.

1.4 Exercises

1.1. Let $T : [0,1] \to [0,1]$ be defined by

$$T(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \in (0,1]; \\ 1, & \text{if } x = 0. \end{cases}$$

Show that for all $x \in [0,1]$ the ω-limit set $\omega(x)$ is non-empty but not forward invariant.

1.2. In Example 1.12 let the masses move in $[0, +\infty)$, and consider the motion with initial positions $q_1(0) < q_2(0)$ and velocities $u_1(0) = 0$ and $u_2(0) = -1$. If $m_2 \geq m_1$, how many collisions among the two balls and among mass m_1 and the wall at $x = 0$ will occur? What happens if $m_2 = 100^n m_1$?

Chapter 2

Continuous-time dynamical systems

2.1 Linear systems

The simplest case to study is that of an ordinary differential equation with linear vector field. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and consider the ordinary differential equation $\underline{\dot{x}} = A\underline{x}$. It is well known that the flow is given by $\phi_t(\underline{x}) = e^{At}\underline{x}$, and the behaviour of the orbits is determined by the eigenvalues of A. We state a result in the case that all the eigenvalues of A are simple, an analogous result holds counting the multiplicities of the eigenvalues and using the Jordan normal form of A.

Theorem 2.1. *Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix with k distinct real eigenvalues $\lambda_1, \dots, \lambda_k$, and $m = \frac{1}{2}(n-k)$ distinct couples of conjugate complex eigenvalues $a_j \pm i\, b_j$. Then there exists an invertible matrix $P \in M(n \times n, \mathbb{R})$ such that*

$$P^{-1}\, A\, P = \Lambda := diag\Big(\lambda_1, \dots, \lambda_k, B_1, \dots, B_m\Big)$$

where

$$B_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \quad \forall\, j = 1, \dots, m,$$

and the flow of the differential equation $\underline{\dot{x}} = A\underline{x}$ is given by

$$\phi_t(\underline{x}) = P\, e^{\Lambda\, t}\, P^{-1}\, \underline{x}$$

where

$$e^{\Lambda\, t} = diag\Big(e^{\lambda_1 t}, \dots, e^{\lambda_k t}, e^{tB_1}, \dots, e^{tB_m}\Big)$$

and

$$e^{tB_j} = e^{a_j t}\begin{pmatrix}\cos(b_j t) & -\sin(b_j t)\\ \sin(b_j t) & \cos(b_j t)\end{pmatrix}, \quad \forall j = 1, \ldots, m.$$

Remark 2.2. Let us consider the case $n = 2, 3$, so that the matrix A can only have multiple real roots. If $n = 2$ the possible Jordan normal form of a matrix A with a double real eigenvalue λ are

$$\Lambda = \mathrm{diag}\Big(\lambda, \lambda\Big) \quad \text{or} \quad \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}.$$

In the non-diagonal case, one writes $\Lambda = \lambda I + N$, where N is the nilpotent matrix

$$N = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

for which $N^2 = 0$. So that[1] $e^{\Lambda t} = e^{\lambda t}\, e^{Nt}$. It follows that

$$e^{\Lambda t} = \mathrm{diag}\Big(e^{\lambda t}, e^{\lambda t}\Big) \quad \text{or} \quad e^{\lambda t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}.$$

Analogously, in the $n = 3$ case, if A has eigenvalues with geometric multiplicities greater than or equal to 2, we are reduced to the previous case. If A has an eigenvalue λ with geometric multiplicity 1 its Jordan normal form is

$$\Lambda = \begin{pmatrix}\lambda & 1 & 0\\ 0 & \lambda & 1\\ 0 & 0 & \lambda\end{pmatrix},$$

and as before we write $\Lambda = \lambda I + N$, where N is a nilpotent matrix such that $N^3 = 0$. Then

$$e^{\Lambda t} = e^{\lambda t}\begin{pmatrix}1 & t & \frac{1}{2}t^2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}.$$

In the case of linear ordinary differential equations it is also particularly simple to find fixed points, periodic orbits, and invariant sets. First, using Definition 1.5 we find

[1] Here we use the fact that the matrices I and N commute.

Proposition 2.3. *The fixed points of the ordinary differential equation $\dot{\underline{x}} = A\underline{x}$ are the points in the kernel of A.*

In particular, the origin $\underline{x}_0 = \underline{0}$ is a fixed point for all A, and the other fixed points come in linear subspaces of $\mathbb{R}^n$. We'll see that the origin plays a special role in characterizing the dynamics of all the non-trivial orbits.

Concerning periodic orbits, it is straightforward from Theorem 2.1 that they can exist only if there is a couple of conjugate complex eigenvalues with null real part. If this is the case, all orbits leaving in the relative eigenspace are periodic, since they are of the form $e^{tB}\underline{x}$ with $a = 0$.

In general, the space $\mathbb{R}^n$ can be written as the direct sum of generalised eigenspaces of A, and according to the asymptotic behaviour of the orbits, it makes sense to consider the following decomposition.

Definition 2.1. Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and let E_λ denote the generalised eigenspace of an eigenvalue λ. We call:
Stable eigenspace of $\underline{0}$ the linear space $E^s(\underline{0})$ defined as

$$E^s(\underline{0}) := \mathrm{Span}\,\{v \in E_\lambda \,:\, \Re(\lambda) < 0\}\ ;$$

Central eigenspace of $\underline{0}$ the linear space $E^c(\underline{0})$ defined as

$$E^c(\underline{0}) := \mathrm{Span}\,\{v \in E_\lambda \,:\, \Re(\lambda) = 0\}\ ;$$

Unstable eigenspace of $\underline{0}$ the linear space $E^u(\underline{0})$ defined as

$$E^u(\underline{0}) := \mathrm{Span}\,\{v \in E_\lambda \,:\, \Re(\lambda) > 0\}\ .$$

Theorem 2.4. *Let $A \in M(n \times n, \mathbb{R})$ be a real $n \times n$ matrix and consider the ordinary differential equation $\dot{\underline{x}} = A\underline{x}$. Then:*

(i) $n = \dim E^s(\underline{0}) + \dim E^c(\underline{0}) + \dim E^u(\underline{0})$*;*

(ii) the eigenspaces $E^s(\underline{0}), E^c(\underline{0}), E^u(\underline{0})$ are invariant;

(iii) the following dynamical characterisation holds:

$$E^s(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to +\infty\}\,;$$

$$E^u(\underline{0}) = \{\underline{x} \in \mathbb{R}^n \,:\, \phi_t(\underline{x}) \to \underline{0} \text{ as } t \to -\infty\}\,.$$

Proof. It is a simple application of Theorem 2.1. □

Remark 2.5. It is interesting to notice that we haven't given a dynamical interpretation for the central eigenspace of $\underline{0}$. The reason is that if $\dim E^c(\underline{0}) \neq 0$ we can find different behaviours for the orbits. Let us consider the simple case $n = \dim E^c(\underline{0}) = 2$ with $\lambda = 0$ being a double eigenvalue. Then there are two possibilities for the matrix A (up to use of the Jordan normal form):

$$A = \operatorname{diag}\Big(0, 0\Big) \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In the first case the flow is the identity, that is $\phi_t(x, y) = (x, y)$ for all $(x, y) \in \mathbb{R}^2$, whereas in the second case the flow is given by $\phi_t(x, y) = (x + ty, y)$ for all $(x, y) \in \mathbb{R}^2$. Using Definition 2.2, in the first case $(0, 0)$ is Lyapunov stable and in the second case it is unstable.

Theorem 2.4 gives the characterisation of the dynamics with respect to the fixed point $\underline{0}$. In particular if $\ker(A) = \{\underline{0}\}$ and $\dim E^c(\underline{0}) = 0$, all orbits converge to $\underline{0}$, either for $t \to +\infty$ or for $t \to -\infty$. If instead the kernel of A consists of a non-trivial linear subspace W with $\dim W = \dim E^c(\underline{0})$, it is easy to see that the dynamics of non-fixed points is determined by that of the points in the space $W^\perp$.

Linear systems in the plane

In the case of linear systems in $\mathbb{R}^2$ it is possible to characterise the dynamical properties of the system without explicitly computing the eigenvalues of the matrix A. We also introduce a terminology for fixed points with different local dynamics.

The nature of the origin $\underline{0} = (0, 0)$ as a fixed point of a system $\dot{\underline{x}} = A\underline{x}$, with $\underline{x} = (x, y) \in \mathbb{R}^2$ is determined by the relation between the determinant and the trace of A. Indeed the characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - \operatorname{tr}(A)\,\lambda + \det(A)\,,$$

so that the eigenvalues are

$$\lambda_\pm = \frac{\operatorname{tr}(A) \pm \sqrt{\operatorname{tr}^2(A) - 4\det(A)}}{2}\,,$$

and we distinguish different cases according to the sign of the determinant of A and of the discriminant $\Delta := \operatorname{tr}^2(A) - 4\det(A)$.

Case 1. $\det(A) > 0$ *and* $\Delta > 0$. The matrix A has two real distinct eigenvalues satisfying $\lambda_+ > \lambda_- > 0$ if $\mathrm{tr}(A) > 0$, and $\lambda_- < \lambda_+ < 0$ if $\mathrm{tr}(A) < 0$.

In both cases the orbits are generalised parabola through $\underline{0}$, at which they are tangent to the line generated by the eigenvector relative to eigenvalue of smallest modulus. If $\mathrm{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable node*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable node*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Note that $\underline{0}$ being a node is an open property since sufficiently small perturbations of A don't change the nature of the origin.

Case 2. $\det(A) > 0$ *and* $\Delta < 0$. The matrix A has a couple of complex conjugate eigenvalues $\lambda_\pm$ with $\Re(\lambda_\pm) = \frac{1}{2}\,\mathrm{tr}(A)$.

If $\mathrm{tr}(A) > 0$ all orbits are spirals out of $\underline{0}$ and they are either clockwise or anti-clockwise according for example to the sign of $\dot{x}$ when $y = 0$. In this case the origin is called an *unstable focus* and $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$ all orbits are spirals into $\underline{0}$ and as before they are either clockwise or anti-clockwise. In this case the origin is called a *stable focus* and $E^s(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) = 0$ all orbits are concentric circles about $\underline{0}$ and again they are either clockwise or anti-clockwise. In this case the origin is called a *center* and $E^c(\underline{0}) = \mathbb{R}^2$.

Notice that $\underline{0}$ being a focus is an open property. Instead $\underline{0}$ being a center is a closed property and arbitrarily small perturbations of A may turn the origin into an unstable or stable focus.

Case 3. $\det(A) > 0$ *and* $\Delta = 0$. The matrix A has one double real eigenvalue $\lambda = \frac{1}{2}\,\mathrm{tr}(A) \neq 0$.

If A is diagonalisable then the orbits lie on straight lines through $\underline{0}$. If $\mathrm{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable star*. We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\mathrm{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable star*. We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

If A is not diagonalisable then we use its Jordan normal form to understand the behaviour of the orbits. The differential equation in normal form reads

$$\begin{cases} \dot{x} = \lambda\, x + y \\ \dot{y} = \lambda y \end{cases}$$

so that there exists an invariant line, which is generated by the eigenvector of A, and the behaviour of the orbits can be found by looking at the sign of the two components of the vector field. If $\text{tr}(A) > 0$, all orbits converge to $\underline{0}$ as $t \to -\infty$, and the origin is called an *unstable improper node.* We also notice that in this case $E^u(\underline{0}) = \mathbb{R}^2$. If $\text{tr}(A) < 0$, all orbits converge to $\underline{0}$ as $t \to +\infty$, and the origin is called a *stable improper node.* We also notice that in this case $E^s(\underline{0}) = \mathbb{R}^2$.

Both $\underline{0}$ being a star and being an improper node are closed properties. An arbitrarily small perturbation can turn the origin into a focus or a node, not changing the stability but the nature of the fixed point.

Case 4. $\det(A) < 0$. The matrix A has a couple of distinct real eigenvalues $\lambda_- < 0 < \lambda_+$.

In this case the orbits are generalised hyperbolae, and the origin is called a *saddle.* It holds $\dim E^u(\underline{0}) = \dim E^s(\underline{0}) = 1$, and none of the orbits outside the eigenspaces approaches the origin as $t \to \pm\infty$. Being a saddle is an open property.

Case 5. $\det(A) = 0$. The matrix A has two real eigenvalues, $\lambda_- = 0$ and $\lambda_+ = \text{tr}(A)$.

If $\text{tr}(A) \neq 0$, then A is diagonalisable and there is a line of fixed points. All the other orbits lie in straight lines which are parallel to the eigenspace of λ_+. If $\text{tr}(A) = 0$ we are reduced to the case of Remark 2.5 up to a change of coordinates, hence either all points are fixed or there is a line of fixed points and all other orbits lie in straight lines which are parallel to the eigenspace of λ_-.

Clearly, the properties of the origin considered in this case are closed and can be changed by arbitrarily small perturbations.

2.2 Stability

Let $\underline{\dot{x}} = F(\underline{x})$ be an ordinary differential equation in $\mathbb{R}^n$ with flow $\phi_t(\cdot)$.

Definition 2.2. A point $\underline{x}$ is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies $d(\phi_t(\underline{x}), \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$.

Remark 2.6. Show that it is necessary to introduce also the notion of orbital stability.

Definition 2.3. A point $\underline{x}$ is *Lyapunov asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that $d(\underline{x}, \underline{y}) < \delta$ implies

$$d(\phi_t(\underline{x}), \phi_t(\underline{y})) \xrightarrow[t \to +\infty]{} 0\,.$$

We call *domain of asymptotic stability of* $\underline{x}$ the set $D(\underline{x})$ of points $\underline{y}$ for which $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$. If $D(\underline{x}) = \mathbb{R}^n$ we say that $\underline{x}$ is *globally Lyapunov asymptotically stable.*

Remark 2.7. If in Definition 2.3 we drop the request that the point $\underline{x}$ is Lyapunov stable, then $\underline{x}$ is called *quasi-asymptotically stable.* In this case there exists a neighbourhood $B_\delta(\underline{x})$ so that $d(\phi_t(\underline{x}), \phi_t(\underline{y})) \to 0$ as $t \to +\infty$ for all $\underline{y} \in B_\delta(\underline{x})$, but the orbits of these points may go arbitrarily far from that of $\underline{x}$ before convergence.

It is particularly important to study the stability of a fixed point $\underline{x}_0$ for which $\phi_t(\underline{x}_0) = \underline{x}_0$ for all t in Definitions 2.2 and 2.3.

Example 2.1 (see [Gl94]). Let us consider the following differential equation in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2+y^2}} \\ \dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2+y^2}} \end{cases}$$

Using polar coordinates (ρ, θ) as shown in Section 2.4 (see (2.8)) with $x = \rho\cos\theta$, $y = \rho\sin\theta$, we are reduced to the equation

$$\begin{cases} \dot{\rho} = \rho(1 - \rho^2) \\ \dot{\theta} = 1 - \cos\theta \end{cases}$$

It is now easy to determine the phase portrait of the equation and deduce that $(x_0, y_0) = (1, 0)$ is a quasi-asymptotically fixed point, but it is not Lyapunov stable.

One first tool to study the stability of a fixed point is to look at the linearisation of the vector field in the point.

Definition 2.4. A fixed point $\underline{x}_0$ of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is called *hyperbolic* if all the eigenvalues of the Jacobian matrix $JF(\underline{x}_0)$ have real part different from zero.

Theorem 2.8 (Hartman-Grobman)**.** *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. Then there exists a neighbourhood $U(\underline{x}_0)$ and a homeomorphism $h : U(\underline{x}_0) \to \mathbb{R}^n$ which sends orbits of the differential equation $\dot{\underline{x}} = F(\underline{x})$ into orbits of the linear differential equation $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$ without changing their direction of time parametrisation[2]. In particular the homeomorphism h leaves invariant the stability properties of the fixed point $\underline{y}_0 = \underline{0}$.*

The proof can be found in Appendix C.

Theorem 2.8 implies that we can characterise a hyperbolic fixed point $\underline{x}_0$ by looking at the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. In particular the qualitative behaviour of the orbits in a neighbourhood of $\underline{x}_0$ coincides with that of the orbits in a neighbourhood of $\underline{y}_0 = \underline{0}$. However, in general, the regularity of h in Theorem 2.8 does not increase by increasing the regularity of a general F. Hence, the "shape" of the orbits may change under the action of h.

The situation is easier in dimension two. If $\underline{x}_0 \in \mathbb{R}^2$ is a hyperbolic fixed point, then $JF(\underline{x}_0)$ is in one of the cases 1-4 excluding case 2 with vanishing trace. If we are not in case 3, the fixed point $\underline{x}_0$ can be characterised like $\underline{y}_0 = \underline{0}$ for $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. Hence we can talk about stable and unstable nodes, stable and unstable foci, and saddles. See [Gl94, Section 5.2].

We now briefly discuss the problem of the regularity of h for $F \in C^\omega$.

Lyapunov functions

Given a real C^1 function $V(\underline{x})$, we introduce the notation $\dot{V}(\underline{x})$ for its derivative along a vector field F. Namely

$$\dot{V}(\underline{x}) := \langle \nabla V(\underline{x}) \, , \, F(\underline{x}) \rangle \tag{2.1}$$

Notice that $\dot{V}(\underline{x}) = \frac{d}{dt} V(\phi_t(\underline{x}))|_{t=0}$.

[2] A formal statement is that, if ϕ_t is the flow of the original system $\dot{\underline{x}} = F(\underline{x})$ and ψ_t is the flow of the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$, then for all $\underline{x} \in U(\underline{x}_0)$ we have $h(\phi_t(\underline{x})) = \psi_t(h(\underline{x}))$ for all $t \in \mathbb{R}$ such that $\phi_t(\underline{x}) \in U(\underline{x}_0)$.

Definition 2.5. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. A C^1 real function $V : U \to \mathbb{R}$ defined in a neighbourhood U of $\underline{x}_0$ is called a *Lyapunov function for* $\underline{x}_0$ if:

(i) $V(\underline{x}) > V(\underline{x}_0)$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$;

(ii) $\dot{V}(\underline{x}) \leq 0$ for all $\underline{x} \in U$.

If the function $V : U \to \mathbb{R}$ satisfies (i) and

(ii)' $\dot{V}(\underline{x}) < 0$ for all $\underline{x} \in U \setminus \{\underline{x}_0\}$,

it is called a *strict Lyapunov function for* $\underline{x}_0$.

Theorem 2.9 (First Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov stable.*

Proof. Let $V : U \to \mathbb{R}$ be the Lyapunov function for $\underline{x}_0$. Given $\varepsilon > 0$ such that $B_\varepsilon(\underline{x}_0) \subset U$, we let

$$m := \min_{\partial B_\varepsilon(\underline{x}_0)} V \quad \text{and} \quad S_m := \{\underline{x} \in B_\varepsilon(\underline{x}_0) \,:\, V(\underline{x}) < m\}$$

By definition $V(\underline{x}_0) < m$, hence $\underline{x}_0 \in S_m$. Moreover by continuity there exists $\delta > 0$ such that $B_\delta(\underline{x}_0) \subset S_m$. We now show that if $\underline{y} \in B_\delta(\underline{x}_0)$ then $\phi_t(\underline{y}) \in B_\varepsilon(\underline{x}_0)$ for all $t \geq 0$.

Condition (ii) in Definition 2.5 implies that $V(\phi_t(\underline{y})) \leq V(\underline{y}) < m$ for all $t \geq 0$. We conclude that if there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{y}) \notin B_\varepsilon(\underline{x}_0)$, then by continuity of the flow there exists $t_1 \in (0, t_0)$ such that $\phi_{t_1}(\underline{y}) \in \partial B_\varepsilon(\underline{x}_0)$. This is a contradiction to the definition of m. □

Theorem 2.10 (Second Lyapunov stability theorem)**.** *Let $\underline{x}_0 \in \mathbb{R}^n$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a strict Lyapunov function for $\underline{x}_0$, then $\underline{x}_0$ is Lyapunov asymptotically stable.*

Proof. Let $V : U \to \mathbb{R}$ be the strict Lyapunov function for $\underline{x}_0$. By Theorem 2.9 the fixed point $\underline{x}_0$ is Lyapunov stable. We now need to show that the domain of asymptotic stability of $\underline{x}_0$ contains a ball $B_\delta(\underline{x}_0)$.

Let us fix $\varepsilon > 0$, and let $\delta > 0$ be such that $d(\underline{x}_0, \underline{y}) < \delta$ implies $d(\underline{x}_0, \phi_t(\underline{y})) < \varepsilon$ for all $t \geq 0$. Hence $\mathcal{O}^+(\underline{y}) \subset B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$, and by Proposition 1.1 we have that $\omega(\underline{y})$ is a non-empty, compact, invariant subset of $B_\varepsilon(\underline{x}_0)$ for all $\underline{y} \in B_\delta(\underline{x}_0)$.

Let us fix $\underline{y} \in B_\delta(\underline{x}_0)$. Condition (ii)' in Definition 2.5 implies that $V(\phi_t(\underline{y}))$ is a decreasing function of t, hence there exists

$$c := \lim_{t\to+\infty} V(\phi_t(\underline{y}))$$

But $V|_{\omega(\underline{y})} \equiv c$ by continuity, in fact for all $\underline{z} \in \omega(\underline{y})$ we have

$$V(\underline{z}) = \lim_{k\to\infty} V(\phi_{t_k}(\underline{y})) = c$$

where $\{t_k\}_k$ is the diverging sequence such that $\phi_{t_k}(\underline{y}) \to z$ as $k \to \infty$. Finally, since $\omega(\underline{y})$ is invariant, we have $V(\phi_t(\underline{z})) = c$ for all t, which by (2.1) implies $\dot V(\underline{z}) = 0$ for all $\underline{z} \in \omega(\underline{y})$. Hence $\omega(\underline{y}) \subset \{\dot V \equiv 0\}$, and by condition (ii)' $\omega(\underline{y}) = \{\underline{x}_0\}$.

We have thus proved that $B_\delta(\underline{x}_0) \subset D(\underline{x}_0)$. □

Corollary 2.11 (La Salle's Invariance Principle)**.** *Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$. If there exists a Lyapunov function for $\underline{x}_0$ defined on a neighbourhood U of $\underline{x}_0$, then for all $\underline{y} \in U$ such that $\mathcal{O}^+(\underline{y})$ is contained in U and is bounded, we have $\omega(\underline{y}) \subseteq \{\dot V \equiv 0\}$.*

Example 2.2. Let us consider the system in $\mathbb{R}^2$ given by

$$\begin{cases} \dot x = y \\ \dot y = -y^3 - x - x^3 \end{cases}$$

The point $(0,0)$ is the only fixed point and it is not hyperbolic. Looking for a Lyapunov function of the form $V(x,y) = ax^2 + bx^4 + cy^2$ one finds $\dot V(x,y) = 2xy(a-c) + 2x^3y(2b-c) - 2cy^4$. Hence

$$V(x,y) = 2x^2 + x^4 + 2y^2$$

is a Lyapunov function for $(0,0)$, with $\{\dot V \equiv 0\} = \{y = 0\}$. Hence V is not a strict Lyapunov function. By Theorem 2.9 we have that $(0,0)$ is Lyapunov stable, and applying Corollary 2.11 we also obtain that there exists $\delta > 0$ such that for all $\underline{y} \in B_\delta((0,0))$ it holds $\omega(\underline{y}) \subset \{y = 0\}$. Moreover, since $\omega(\underline{y})$ is an invariant set and the only invariant subset of $\{y = 0\}$ is $\{(0,0)\}$, we have proved that $(0,0)$ is asymptotically stable.

Theorem 2.12 (Bounding functions)**.** *Let F be a vector field in $\mathbb{R}^n$, and assume that there exist a C^1 real function $V : \mathbb{R}^n \to \mathbb{R}$, a compact set $G \subset \mathbb{R}^n$ and $k \in \mathbb{R}$ such that: (a) $G \subset V_k := \{V < k\}$; (b) there exists $\delta > 0$ such that $\dot V(\underline{x}) \leq -\delta$ for all $\underline{x} \in \mathbb{R}^n \setminus G$. Then for all $\underline{x} \in \mathbb{R}^n$ there exists $t_0 \geq 0$ such that $\phi_t(\underline{x}) \in V_k$ for all $t > t_0$.*

Proof. If $\underline{x} \in V_k$ we are done, since by assumption (b) $\dot{V}|_{\partial V_k} < 0$, and we can choose $t_0 = 0$. If $\underline{x} \notin V_k$ and $\phi_t(\underline{x}) \notin V_k$ for all $t > 0$

$$V(\phi_t(\underline{x})) - V(\underline{x}) = \int_0^t \frac{d}{ds} V(\phi_s(\underline{x}))\, ds = \int_0^t \dot{V}(\phi_s(\underline{x}))\, ds \leq -\delta t$$

which implies $V(\phi_t(\underline{x})) < k$ for $t > \frac{V(\underline{x})-k}{\delta}$. Hence we find a contradiction, and we have thus proved that there exists $t_0 > 0$ such that $\phi_{t_0}(\underline{x}) \in V_k$, and as before this implies that $\phi_t(\underline{x}) \in V_k$ for all $t \geq t_0$. □

Example 2.3 (Lorenz equations). Let us consider the system in $\mathbb{R}^3$ given by

$$\begin{cases} \dot{x} = \sigma(-x + y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

with σ, r, b positive constants. We can apply Theorem 2.12 with

$$G = \left\{ (x, y, z) \in \mathbb{R}^3 \, : \, rx^2 + y^2 + b(z - r)^2 < 2br^2 \right\}$$

$$V(x, y, z) = \frac{1}{2} \Big(rx^2 + \sigma y^2 + \sigma (z - 2r)^2 \Big)$$

and $\delta = \sigma b r^2$.

Using the theory of Lyapunov functions we now give a proof of the asymptotic stability of *sinks*, i.e. hyperbolic fixed points of a C^1 vector field with all eigenvalues of the Jacobian matrix of the field with negative real part.

Corollary 2.13. *Let $\underline{x}_0$ be a hyperbolic fixed point of a C^1 vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, and assume that all the eigenvalues of $JF(\underline{x}_0)$ have negative real part. Then $\underline{x}_0$ is asymptotically stable.*

Proof. Let's assume without loss of generality that $\underline{x}_0 = \underline{0}$, then the vector field F satisfies $F(\underline{0}) = \underline{0}$ and can be written as

$$F(\underline{x}) = JF(\underline{0})\underline{x} + G(\underline{x})$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function satisfying $G(\underline{0}) = \underline{0}$ and $JG(\underline{0}) = 0$.

Let $\lambda_1, \ldots, \lambda_k$ be the, not necessarily distinct, negative real eigenvalues of $JF(\underline{0})$, and let $a_j \pm i b_j$, with $j = 1, \ldots, \frac{1}{2}(n-k)$, be the, not necessarily distinct, couples of conjugate complex eigenvalues with $a_j < 0$. For simplicity we also assume that $JF(\underline{0})$ is written in Jordan normal form, therefore

$$JF(\underline{0}) = diag\Big(\Lambda_1, \dots, \Lambda_h, B_1, \dots B_m\Big)$$

where the Λ_j's are the Jordan blocks relative to the real eigenvalues, and the B_j's are the Jordan blocks relative to the complex eigenvalues.

Let us consider the following change of variables. For $\varepsilon > 0$ let $\underline{y} = (y_1, \dots, y_n)$ be defined as follows:

- if $(x_m, \dots, x_{m+s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block Λ_j, we let $y_{m+\ell} := \varepsilon^{-\ell} x_{m+\ell}$ for $\ell = 0, \dots, s-1$;
- if $(x_p, \dots, x_{p+2s-1})$ are the components of $\underline{x}$ corresponding to a Jordan block B_j, we let $y_{p+2\ell} := \varepsilon^{-\ell} x_{p+2\ell}$ and $y_{p+2\ell+1} := \varepsilon^{-\ell} x_{p+2\ell+1}$, for $\ell = 0, \dots, s-1$.

Then it is a standard computation to verify that $\underline{y}$ satisfies the ODE

$$\dot{\underline{y}} = A_\varepsilon \underline{y} + \tilde{G}(\underline{y}),$$

with $\tilde{G}(\underline{0}) = \underline{0}$ and $J\tilde{G}(\underline{0}) = 0$ and

$$A_\varepsilon = diag\Big(\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_h, \tilde{B}_1, \dots \tilde{B}_m\Big),$$

where

$$\tilde{\Lambda}_j = \begin{pmatrix} \lambda_j & \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \lambda_j & \varepsilon & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \lambda_j & \varepsilon & 0 \\ 0 & \dots & \dots & 0 & \lambda_j & \varepsilon \\ 0 & \dots & \dots & \dots & 0 & \lambda_j \end{pmatrix}$$

and

$$\tilde{B}_j = \begin{pmatrix} R_j & \varepsilon I_2 & 0 & \dots & 0 & 0 \\ 0 & R_j & \varepsilon I_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & R_j & \varepsilon I_2 & 0 \\ 0 & \dots & \dots & 0 & R_j & \varepsilon I_2 \\ 0 & \dots & \dots & \dots & 0 & R_j \end{pmatrix}, \quad \text{with } R_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$$

and I_2 the 2×2 identity matrix.

We now show that $V(\underline{y}) = \sum_{i=1}^n y_i^2$ is a strict Lyapunov function for $\underline{0}$. It is enough to study the derivative $\dot{V}(\underline{y}) = 2\sum_{i=1}^n y_i \dot{y}_i$.

If $(y_m, \ldots, y_{m+s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{\Lambda}_j$ we have

$$\sum_{\ell=0}^{s-1} y_{m+\ell}\,\dot{y}_{m+\ell} = \sum_{\ell=0}^{s-2} y_{m+\ell}(\lambda_j y_{m+\ell} + \varepsilon y_{m+\ell+1}) + \lambda_j y_{m+s-1}^2 + O(|\underline{y}|^3) =$$

$$\le \lambda_j \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + \frac{\varepsilon}{2}(y_m^2 + y_{m+s-1}^2) + \varepsilon \sum_{\ell=1}^{s-2} y_{m+\ell}^2 + O(|\underline{y}|^3) \le$$

$$\le (\lambda_j + \varepsilon) \sum_{\ell=0}^{s-1} y_{m+\ell}^2 + O(|\underline{y}|^3).$$

With an analogous argument, if $(y_m, \ldots, y_{m+2s-1})$ are the components of $\underline{y}$ corresponding to a Jordan block $\tilde{B}_j$ we have

$$\sum_{\ell=0}^{s-1}(y_{m+2\ell}\,\dot{y}_{m+2\ell} + y_{m+2\ell+1}\,\dot{y}_{m+2\ell+1}) =$$

$$= \sum_{\ell=0}^{s-2} y_{m+2\ell}\,(a_j y_{m+2\ell} - b_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+2}) +$$

$$+ \sum_{\ell=0}^{s-2} y_{m+2\ell+1}\,(b_j y_{m+2\ell} + a_j y_{m+2\ell+1} + \varepsilon y_{m+2\ell+3}) +$$

$$+ y_{m+2s-2}\,(a_j y_{m+2s-2} - b_j y_{m+2s-1}) + y_{m+2s-1}\,(b_j y_{m+2s-2} + a_j y_{m+2s-1}) +$$

$$+ O(|\underline{y}|^3) =$$

$$= a_j \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + \varepsilon \sum_{\ell=0}^{s-2} \Big(y_{m+2\ell}\,y_{m+2\ell+2} + y_{m+2\ell+1}\,y_{m+2\ell+3}\Big) +$$

$$+ O(|\underline{y}|^3) \le$$

$$\le (a_j + \varepsilon) \sum_{\ell=0}^{s-1}(y_{m+2\ell}^2 + y_{m+2\ell+1}^2) + O(|\underline{y}|^3).$$

If we fix $\varepsilon > 0$ such that $(\lambda_j + \varepsilon) < 0$ and $(a_j + \varepsilon) < 0$ for all eigenvalues of $JF(\underline{0})$, letting $\mu \in \mathbb{R}^-$ satisfy $(\lambda_j + \varepsilon) \leq \mu < 0$ and $(a_j + \varepsilon) \leq \mu < 0$ for all j, we have proved that

$$\dot{V}(\underline{y}) \leq 2\mu|\underline{y}|^2 + O(|\underline{y}|^3).$$

We need to show that there exists $\delta > 0$ such that $\dot{V}(\underline{y}) < 0$ for all $\underline{y} \in B_\delta(\underline{0})$ and $\underline{y} \neq \underline{0}$. By definition of $O(\cdot)$ functions, there exist $c > 0$ and $\tilde{\delta} > 0$ such that

$$O(|\underline{y}|^3) \leq c|\underline{y}|^3\,, \quad \forall\, \underline{y} \in B_{\tilde{\delta}}(\underline{0}).$$

If we choose $\delta = \min\{-\frac{2\mu}{c}, \tilde{\delta}\}$ it follows

$$\dot{V}(\underline{y}) \leq 2\mu|\underline{y}|^2 + c|\underline{y}|^3 = |\underline{y}|^2(2\mu + c|\underline{y}|) < 0, \quad \forall\, \underline{y} \in B_\delta(\underline{0}) \setminus \{\underline{0}\},$$

and the proof is finished. □

2.3 Integrals of motion and invariant sets

Conservative systems and first integrals

Definition 2.6. A C^1 function $I : \mathbb{R}^n \to \mathbb{R}$ is a *first integral* for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}(\underline{x}) = 0$ for all $\underline{x} \in \mathbb{R}^n$, with $\dot{I}(\underline{x})$ defined as in (2.1).

If $I : \mathbb{R}^n \to \mathbb{R}$ is a first integral for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, then its level sets are invariant for the differential equation $\dot{\underline{x}} = F(\underline{x})$, so that in particular orbits of $\dot{\underline{x}} = F(\underline{x})$ lie in the level sets of I.

An important example of differential equations with a first integral are Hamiltonian systems with Hamiltonian function independent of time.

Definition 2.7. Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a C^1 function and use the notation $(\underline{x}, \underline{y})$ for points in $\mathbb{R}^{2n}$, with $\underline{x}, \underline{y} \in \mathbb{R}^n$. The *Hamiltonian vector field associated to* H is $F_H : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ given for $i = 1, \ldots, n$, by $(F_H)_i = \partial H / \partial y_i$ and $(F_H)_{(n+i)} = -\partial H / \partial x_i$, and H is called the *Hamiltonian function* of the field. The system of differential equations in $\mathbb{R}^{2n}$ with field F_H is called the *Hamiltonian system of* H.

A particular case are conservative mechanical systems with one degree of freedom, systems which describe for example the motion in $\mathbb{R}$ of a point of mass m under conservative forces. In this case the Hamiltonian function has the form

$$H : \mathbb{R}^2 \to \mathbb{R}\,, \quad H(x,y) = \frac{1}{2m}\, y^2 + W(x) \tag{2.2}$$

where $W(x) \in C^1$ is the potential energy of the system. We recall that in this case the Hamiltonian system associated to H is

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y}(x,y) = \frac{1}{m}\, y \\ \dot{y} = -\frac{\partial H}{\partial x}(x,y) = -W'(x) \end{cases}$$

and corresponds to the second-order differential equation $m\ddot{x} = -W'(x)$.

Proposition 2.14. *A C^1 function H is a first integral for the Hamiltonian vector field F_H.*

Proof. A simple computation gives

$$\dot{H} = \langle \nabla H\,,\, F_H \rangle = \sum_{i=1}^{n} \Big(\frac{\partial H}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial H}{\partial y_i} \frac{\partial H}{\partial x_i} \Big) \equiv 0.$$

□

Theorem 2.15 (Liouville theorem)**.** *A Hamiltonian system in $\mathbb{R}^{2n}$ with C^2 Hamiltonian function H preserves the $2n$-dimensional Lebesgue measure of the sets.*

Proof. For $A \subset \mathbb{R}^{2n}$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the $2n$-dimensional Lebesgue measure. Then

$$m(\phi_t(A)) = \int_{\phi_t(A)} 1\, dm = \int_A |\det(J\phi_t)|\, dm.$$

The variation equation of a differential equation shows that $J\phi_t$ satisfies the Cauchy problem

$$\begin{cases} \frac{d}{dt} J\phi_t(\underline{x}) = JF_H(\phi_t(\underline{x}))\, J\phi_t(\underline{x}) \\ J\phi_t(\underline{x})|_{t=0} = I \end{cases}$$

where I is the identity matrix. The solution to the previous Cauchy problem is then

$$J\phi_t(\underline{x}) = \exp\Big(\int_0^t JF_H(\phi_s(\underline{x}))\, ds\Big)\, I,$$

and using the identity $\det(\exp(M)) = \exp(\mathrm{tr}(M))$, valid for any finite square matrix M, we obtain

$$\det(J\phi_t(\underline{x})) = \exp\Big(\int_0^t \mathrm{tr}(JF_H(\phi_s(\underline{x})))\, ds\Big).$$

Then

$$m(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F_H)(\phi_s(\underline{x}))\, ds \right) dm.$$

Since

$$\mathrm{div}(F_H) = \sum_{i=1}^n \Big(\frac{\partial^2 H}{\partial x_i \partial y_i} - \frac{\partial^2 H}{\partial y_i \partial x_i} \Big) \equiv 0,$$

it follows that

$$m(\phi_t(A)) = m(A)\,, \quad \forall\, t \in \mathbb{R}$$

and the proof is finished. □

Corollary 2.16. *A Hamiltonian system in $\mathbb{R}^{2n}$ cannot have fixed points which are sinks or sources.*

Let us consider mechanical Hamiltonian systems with one degree of freedom with Hamiltonian function $H(x, y)$ as in (2.2). Applying the general theory of the previous sections and the results in this section, one can easily prove the following characterisation of the fixed points.

Proposition 2.17. *Let $H : \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function written as in* (2.2). *Then the fixed points of the associated Hamiltonian system are of the form* $(x_0, 0)$ *with* $W'(x_0) = 0$.
If $W''(x_0) < 0$ *then* $(x_0, 0)$ *is a hyperbolic fixed point of saddle type, if* $W''(x_0) > 0$ *it is not hyperbolic and it is a center.*
If $W''(x_0) = 0$ *the point* $(x_0, 0)$ *is not hyperbolic and one needs to use the level sets of* $H(x, y)$ *to study the dynamics in a neighbourhood of the point.*

Example 2.4. The Hamiltonian function of a pendulum of mass m and length ℓ in a vertical gravitational field with potential energy $W(h) = mgh$ is

$$H(x,y) = \frac{1}{2m\ell^2}\, y^2 + mg\ell(1 - \cos x)\,,$$

Consider the motion of this pendulum in presence of a constant friction given by $-\mu\, y$, with $\mu \geq 0$.

Example 2.5. Study the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 - \mu\, y \end{cases}$$

with $\mu \in \mathbb{R}$.

Invariant sets

It is in general difficult to find explicit expressions for invariant sets. However, there are particular easy situations. For example, given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with $F = (F_1, \ldots, F_n)$, if there exists $c \in \mathbb{R}$ such that $F_i(x_1, \ldots, x_{i-1}, c, x_{i+1}, \ldots, x_n) = 0$ for all $x_j \in \mathbb{R}$ with $j \neq i$, then the hyperplane $\{x_i = c\}$ is an invariant set. This can be proved by the following method.

Proposition 2.18. *Let $I : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function and for $c \in \mathbb{R}$ let $I_c := \{I(\underline{x}) = c\}$ be a non-empty level set of I such that $\nabla I|_{I_c} \not\equiv 0$. The level set I_c is invariant for a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ if $\dot{I}|_{I_c} \equiv 0$.*

Proof. Let $\underline{x}_0 \in I_c$ such that $\nabla I(\underline{x}_0) \neq 0$. Then there exists a local differentiable change of coordinates $\underline{y} = h(\underline{x})$ such that in a neighbourhood $U(\underline{x}_0)$ we have $I_c \cap U = \{y_n = 0\}$ and let $\underline{x}_0 = (\tilde{\underline{y}}_0, 0)$ with $\tilde{\underline{y}}_0 \in \mathbb{R}^{n-1}$. Hence, in these new coordinates $\nabla I \in \mathrm{Span}\{(0, \ldots, 0, 1)\}$ in U.

Then, from $\dot{I}|_{I_c} \equiv 0$, we have that $F_n|_U \equiv 0$. Let $\tilde{F} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be defined as $\tilde{F}(y_1, \ldots, y_{n-1}) = (F_1(y_1, \ldots, y_{n-1}, 0), \ldots, F_{n-1}(y_1, \ldots, y_{n-1}, 0))$.

Then by the local uniqueness of the solutions to the system $\dot{\underline{x}} = F(\underline{x})$, the solution with initial condition in $\underline{x}_0$ coincides in U with $(\tilde{\phi}_t(\tilde{\underline{y}}_0), 0)$, where $\tilde{\phi}_t$ is the flow of the system $\dot{\tilde{\underline{y}}} = \tilde{F}(\tilde{\underline{y}})$. Hence, the solution is in I_c. This proves the invariance of I_c. □

Example 2.6. Given the system

$$\begin{cases} \dot{x} = x^2 - y - 1 \\ \dot{y} = (x-2)y \end{cases}$$

the lines $y = 0$, $y = x + 1$ and $y = 3x - 3$ are invariant sets.

Stable and unstable manifolds

An important example of invariant sets is given by the *stable* and *unstable manifolds* of a hyperbolic fixed point.

Definition 2.8. Let $\underline{x}_0$ be a fixed point of a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$, and let U be a neighbourhood of $\underline{x}_0$. The *local stable manifold* $W^s_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^s_{loc}(\underline{x}_0) := \{\underline{x} \in U \,:\, \phi_t(\underline{x}) \in U \text{ for all } t \geq 0,\ \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to +\infty\}$$

Analogously, the *local unstable manifold* $W^u_{loc}(\underline{x}_0)$ *of* $\underline{x}_0$ *in* U is the set

$$W^u_{loc}(\underline{x}_0) := \{\underline{x} \in U \,:\, \phi_t(\underline{x}) \in U \text{ for all } t \leq 0,\ \phi_t(\underline{x}) \to \underline{x}_0 \text{ as } t \to -\infty\}$$

Theorem 2.19 (Stable and unstable manifolds)**.** *Let $\underline{x}_0$ be a fixed point of a C^k, $k \geq 1$, vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$. Let's assume that $\underline{x}_0$ is hyperbolic and let $E^s(\underline{0})$ and $E^u(\underline{0})$ be the stable and unstable eigenspaces associated to the linear system $\dot{\underline{y}} = JF(\underline{x}_0)\underline{y}$. Then there exists $\varepsilon > 0$ such that there exist local stable and unstable manifolds, $W^s_{loc}(\underline{x}_0)$ and $W^u_{loc}(\underline{x}_0)$, of $\underline{x}_0$ in $B_\varepsilon(\underline{x}_0)$ with the following properties:*

(i) *$W^s_{loc}(\underline{x}_0)$ and $W^u_{loc}(\underline{x}_0)$ are unique in $B_\varepsilon(\underline{x}_0)$;*

(ii) *$W^s_{loc}(\underline{x}_0)$ is forward invariant, and $W^u_{loc}(\underline{x}_0)$ is backward invariant;*

(iii) *$W^s_{loc}(\underline{x}_0)$ and $W^u_{loc}(\underline{x}_0)$ are C^k manifolds,* $\dim W^s_{loc}(\underline{x}_0) = \dim E^s(\underline{0})$ *and* $\dim W^u_{loc}(\underline{x}_0) = \dim E^u(\underline{0})$*;*

(iv) *$W^s_{loc}(\underline{x}_0)$ is tangential to $\underline{x}_0 + E^s(\underline{0})$ at $\underline{x}_0$, and $W^u_{loc}(\underline{x}_0)$ is tangential to $\underline{x}_0 + E^u(\underline{0})$ at $\underline{x}_0$.*

Proof of Theorem 2.19 in $\mathbb{R}^2$ (see [HSD]). Without loss of generality, let's assume that the fixed point is $(x_0, y_0) = (0,0)$. If $\dim E^s = 2$ or $\dim E^u = 2$, the proof is trivial since in these cases either $W^s_{loc}(0,0)$ or $W^u_{loc}(0,0)$, respectively, coincide with a ball around $(0,0)$ and the properties of the statement follow from the Hartman-Grobman Theorem 2.8.

The interesting case is when $\dim E^s = \dim E^u = 1$ and $(0,0)$ is a saddle. Up to a change of variables, we can assume that the system is written as

$$\begin{cases} \dot{x} = -\lambda\, x + f(x,y) \\ \dot{y} = \mu\, y + g(x,y) \end{cases} \tag{2.3}$$

with $\lambda, \mu > 0$, $f, g \in C^k$ with $f(0,0) = g(0,0) = 0$, and $f, g = O(x^2 + y^2)$ if $k \geq 2$, and $f, g, \partial_x f, \partial_y f, \partial_x g, \partial_y g = o(\sqrt{x^2+y^2})$ if $k = 1$. Hence, $E^s = \mathrm{Span}\{(1,0)\}$ and $E^u = \mathrm{Span}\{(0,1)\}$.

We give the proof for the local stable manifold, it follows analogously for the local unstable one. For any $\varepsilon > 0$ and $M > 1$, introduce the following notations:

$$\begin{aligned} D_\varepsilon &:= \{|x| \leq \varepsilon,\ |y| \leq \varepsilon\}\,, \quad & C_M &:= \{|x| \geq M\,|y|\}\,, \\ S^\pm_\varepsilon &:= C_M \cap \{x = \pm\varepsilon\}\,, \quad & C^\pm_M &:= C_M \cap \{x \gtrless 0\}\,. \end{aligned} \tag{2.4}$$

The proof is divided into different steps.

Step I. There exists $\varepsilon_0 > 0$ such that for all $M > 1$ we have $\dot{x}|_{D_\varepsilon \cap C^\pm_M} \lessgtr 0$. By assumption, there exists $\varepsilon > 0$ such that

$$|f(x,y)| \leq \frac{\lambda}{2\sqrt{2}}\sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then, on $D_\varepsilon \cap C^+_M$, we have

$$\begin{aligned} \dot{x} &= -\lambda\, x + f(x,y) \leq -\lambda\, x + \frac{\lambda}{2\sqrt{2}}\sqrt{x^2+y^2} \leq \\ &\leq -\lambda\, x + \frac{\lambda}{2\sqrt{2}}\sqrt{x^2\left(1+\frac{1}{M^2}\right)} \leq x\left(-\lambda + \frac{\lambda}{2}\right) = -\frac{\lambda}{2}\, x < 0. \end{aligned}$$

Similarly, on $D_\varepsilon \cap C^-_M$, we have

$$\begin{aligned} \dot{x} &= -\lambda\, x + f(x,y) \geq -\lambda\, x - \frac{\lambda}{2\sqrt{2}}\sqrt{x^2+y^2} \geq \\ &\geq -\lambda\, x - \frac{\lambda}{2\sqrt{2}}\sqrt{x^2\left(1+\frac{1}{M^2}\right)} \geq |x|\left(\lambda - \frac{\lambda}{2}\right) = \frac{\lambda}{2}\,|x| > 0. \end{aligned}$$

Step II. For any $M > 1$, there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that for $\varepsilon \in (0, \varepsilon_1)$ on the boundary of $D_\varepsilon \cap C_M$ the field F points towards the outside of C_M.
First of all, we consider $\varepsilon \in (0, \varepsilon_0)$ with ε_0 from Step I. Let us study the case $x, y > 0$. We have $\dot{x}|_{\partial(D_\varepsilon \cap C_M^+)} < 0$. It is then enough to prove that $\dot{y}|_{\partial(D_\varepsilon \cap C_M^+)} > 0$. Fixed $M > 1$, by assumption, there exists $\varepsilon_1 < \varepsilon_0$ such that for $\varepsilon \in (0, \varepsilon_1)$

$$|g(x,y)| \leq \frac{\mu}{2\sqrt{1+M^2}} \sqrt{x^2+y^2}, \quad \forall (x,y) \in D_\varepsilon.$$

Then, on $\partial(D_\varepsilon \cap C_M^+)$, we have

$$\begin{aligned}\dot{y} &= \mu\, y + g(x,y) \geq \mu\, y - \frac{\mu}{2\sqrt{1+M^2}} \sqrt{x^2+y^2} = \\ &= \mu\, y - \frac{\mu}{2\sqrt{1+M^2}} \sqrt{y^2(M^2+1)} = y\left(\mu - \frac{\mu}{2}\right) = \frac{\mu}{2}\, y > 0.\end{aligned}$$

The other cases follow analogously.

Step III. For $\varepsilon \in (0, \varepsilon_1)$, on S_ε^+ there exist non-empty open intervals I_+ and I_- such that for all $(x_0, y_0) \in I_\pm$ the orbit $\phi_t(x_0, y_0)$ intersects $\partial(D_\varepsilon \cap C_M^+) \cap \{y \gtrless 0\}$.
The existence and the properties of the intervals I_+ and I_- follow from Steps I and II, and from the local uniqueness and the continuity with respect to the initial conditions of the solutions to (2.3).

Step IV. There exists $\varepsilon_2 < \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2)$ The set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ consists of a single point $(\varepsilon, \bar{y}_+(\varepsilon))$.
By the properties of the solutions to (2.3), there exist y_1, y_2 such that

$$S_\varepsilon^+ \setminus (I_+ \cup I_-) = \{(\varepsilon, y) \, : y \in [y_1, y_2]\}.$$

We need to show that $y_1 = y_2 = \bar{y}_+$.

Let's assume that $y_1 < y_2$. It is known that multiplying a vector field $F(x,y)$ by a non-vanishing function $h(x,y)$, the orbits of the system do not change but only their time-parametrisation is affected. Let $h(x,y) = 1/(\lambda - f(x,y)/x)$ in $D_\varepsilon \cap C_M$. Then the system in $D_\varepsilon \cap C_M$ becomes

$$\begin{cases} \dot{x} = -x \\ \dot{y} = \frac{\mu\, y + g(x,y)}{\lambda - \frac{f(x,y)}{x}} = \frac{\mu}{\lambda}\, y + \tilde{g}(x,y) \end{cases} \tag{2.5}$$

with $\tilde{g}, \partial_y \tilde{g} = o(\sqrt{x^2+y^2})$. There exists $\varepsilon_2 < \varepsilon_1$ such that for all $M > 1$ and all $\varepsilon \in (0, \varepsilon_2)$ we have

$$\left| \frac{\partial \tilde{g}}{\partial y}(x,y) \right| \leq \frac{\mu}{2\lambda\sqrt{2}} \sqrt{x^2+y^2}\,, \quad \forall\,(x,y) \in D_\varepsilon.$$

Then the solutions to (2.5) with initial condition (ε, y) are of the form $(\varepsilon\, e^{-t}, y(t))$. Hence, we can compute the vertical distance between the orbits $\phi_t(\varepsilon, y_1)$ and $\phi_t(\varepsilon, y_2)$ by computing the distance $y_2(t) - y_1(t)$ of the second components of the solutions to (2.5) with initial conditions (ε, y_1) and (ε, y_2). We have

$$\begin{aligned} \frac{d}{dt}(y_2(t) - y_1(t)) &= \frac{\mu}{\lambda}\, y_2(t) + \tilde{g}(\varepsilon\, e^{-t}, y_2(t)) - \frac{\mu}{\lambda}\, y_1(t) + \tilde{g}(\varepsilon\, e^{-t}, y_1(t)) = \\ &= \frac{\mu}{\lambda}\,(y_2(t) - y_1(t)) + \frac{\partial \tilde{g}}{\partial y}(\varepsilon\, e^{-t}, \xi(t))\,(y_2(t) - y_1(t)) \geq \\ &\geq (y_2(t) - y_1(t)) \left(\frac{\mu}{\lambda} - \frac{\mu}{2\lambda\sqrt{2}}\,\varepsilon\, e^{-t} \sqrt{1 + \frac{1}{M^2}} \right) \geq \\ &\geq \frac{\mu}{2\lambda}\,(y_2(t) - y_1(t)). \end{aligned}$$

Hence, $(y_2(t) - y_1(t)) \to +\infty$, which contradicts that the orbits of the set $S_\varepsilon^+ \setminus (I_+ \cup I_-)$ are forward asymptotic to $(0,0)$. We have thus proved that $y_1 = y_2 = \bar{y}_+$.

Conclusion part I.
By Steps I-IV, for all $M > 1$ there exists $\varepsilon_2 = \varepsilon_2(M) > 0$ such that for all $\varepsilon \in (0, \varepsilon_2)$ we obtain the existence of a unique point $(\varepsilon, \bar{y}_+(\varepsilon))$ in S_ε^+ whose orbit is forward asymptotic to $(0,0)$. Therefore, fixing a $\bar{M} > 1$ the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^+$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ is given by the forward orbit of the point $(\varepsilon_2(\bar{M}), \bar{y}_+(\varepsilon_2(\bar{M})))$. An analogous argument shows the existence of the local stable manifold $W^s_{loc}(0,0)$ in $D_\varepsilon \cap C_M^-$ for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$.

This shows the uniqueness of $W^s_{loc}(0,0)$, its forward invariance, its regularity since the orbits of a system inherit the regularity of the vector field, and that its dimension is 1. It remains to prove that $W^s_{loc}(0,0)$ is tangent at $(0,0)$ to $E^s = \mathrm{Span}\{(1,0)\}$, that is to the x-axis.

Step V. Fixing a $\bar{M} > 1$, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the orbit $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$ has vanishing angular coefficient as $t \to +\infty$.

Let $(x_\varepsilon(t), y_\varepsilon(t))$ denote the two components of $\phi_t(\varepsilon, \bar{y}_+(\varepsilon))$. We need to show that $y_\varepsilon(t)/x_\varepsilon(t) \to 0$ as $t \to +\infty$.

By the local uniqueness of the solutions to (2.3), for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$ the point $(\varepsilon, \bar{y}_+(\varepsilon))$ is in the orbit of $(\varepsilon', \bar{y}_+(\varepsilon'))$ for all $\varepsilon' \in (\varepsilon, \varepsilon_2(\bar{M}))$. This shows that, for all $\varepsilon \in (0, \varepsilon_2(\bar{M}))$, for all $t \geq 0$ there exists $\tilde{\varepsilon}(t) < \varepsilon$ such that $(x_\varepsilon(t), y_\varepsilon(t)) = (\tilde{\varepsilon}(t), \bar{y}_+(\tilde{\varepsilon}(t)))$. Hence, as $t \to +\infty$ we have $\tilde{\varepsilon}(t) \to 0^+$ so that $(x_\varepsilon(t), y_\varepsilon(t))$ is in $\cap_{\bar{M} \leq M \leq \tilde{M}(t)} C_M^+$ for some $\tilde{M}(t) \to +\infty$. This shows that $y_\varepsilon(t)/x_\varepsilon(t) \leq 1/\tilde{M}(t) \to 0^+$ as $t \to +\infty$.

Conclusion part II.
Step V concludes the proof of the theorem. □

Given a hyperbolic fixed point $\underline{x}_0$, one can introduce a notion of *global* stable and unstable manifolds. However, these sets in general have weaker properties than the local counterparts.

Definition 2.9. Let $\underline{x}_0$ be a hyperbolic fixed point of a C^k, $k \geq 1$, vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ with flow $\phi_t(\cdot)$. The *global stable* and *unstable manifolds* of $\underline{x}_0$ are defined as

$$W^s(\underline{x}_0) := \bigcup_{t \leq 0} \phi_t(W^s_{loc}(\underline{x}_0)), \qquad W^u(\underline{x}_0) := \bigcup_{t \geq 0} \phi_t(W^u_{loc}(\underline{x}_0)), \tag{2.6}$$

where $W^{s,u}_{loc}(\underline{x}_0)$ are the local manifolds in $B_\varepsilon(\underline{x}_0)$ for some $\varepsilon > 0$.

It is interesting to analyse the possible intersection of the global stable and unstable manifolds. By the local uniqueness of the solutions to an ODE, the two global manifolds cannot intersect transversally. In $\mathbb{R}^2$ they can coincide or end up at another saddle fixed point, giving rise to a homoclinic or two heteroclinic orbits respectively. In $\mathbb{R}^n$ with $n \geq 3$ more interesting phenomena occurs, and some imply the existence of "chaotic" phenomena (see Section 3.4).

2.4 Motion in the plane and periodic orbits

In this section we consider a system of differential equations in $\mathbb{R}^2$

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \tag{2.7}$$

with C^k, $k \geq 1$, functions $f, g : \mathbb{R}^2 \to \mathbb{R}$. We discuss methods to study the phase space of (2.7) which work in two dimensions.

Polar coordinates

In $\mathbb{R}^2$, it is sometimes easier to study the phase space of a system when using polar coordinates. Let

$$\Omega := \left\{(\rho,\theta) \in \mathbb{R}^2 \,:\, \rho > 0\,,\, 0 \leq \theta \leq 2\pi\right\} / (\{\theta = 0\} = \{\theta = 2\pi\}),$$

that is Ω is a strip in the plane with the upper and the lower boundary identified, hence it is an open cylinder. The map $\psi : \Omega \to \mathbb{R}^2$, $(x,y) = \psi(\rho,\theta)$, with

$$\begin{cases} x(\rho,\theta) = \rho\, \cos\theta \\ y(\rho,\theta) = \rho\, \sin\theta \end{cases}$$

is a diffeomorphism from Ω to $\mathbb{R}^2$, with Jacobian $\det J\psi(\rho,\theta) = \rho$. We can then use ψ and its inverse to push a vector field $F(x,y) = (f(x,y), g(x,y))$ back to a vector field on Ω. An easy computation shows that the system (2.7) when written in polar coordinates reads

$$\begin{cases} \dot{\rho} = f(\rho\cos\theta, \rho\sin\theta)\, \cos\theta + g(\rho\cos\theta, \rho\sin\theta)\, \sin\theta \\ \dot{\theta} = \frac{g(\rho\cos\theta,\rho\sin\theta)\, \cos\theta - f(\rho\cos\theta,\rho\sin\theta)\, \sin\theta}{\rho} \end{cases} \tag{2.8}$$

for all $(\rho,\theta) \in \Omega$. In general one should not expect that the vector field in polar coordinates can be continuously extended to the boundary $\{\rho = 0\}$ of Ω. This is true only under particular conditions on the functions f, g.

An important application of the use of polar coordinates is the identification of circular periodic orbits. Using Proposition 2.18 one can show that

Proposition 2.20. *If there exists $\rho_0 > 0$ such that*

$$f(\rho_0\cos\theta, \rho_0\sin\theta)\, \cos\theta + g(\rho_0\cos\theta, \rho_0\sin\theta)\, \sin\theta = 0\,, \quad \forall\,\theta \in [0, 2\pi]$$

and $\dot{\theta} \neq 0$ *for all* $(\rho, \theta) \in \{\rho = \rho_0\}$, *then the set*

$$\Gamma = \{\rho = \rho_0\} = \left\{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = \rho_0^2\right\}$$

is a periodic orbit.

Isoclines

Here we introduce a method to find an analytic expression for the orbits of (2.7) in special situations.

Proposition 2.21. *Let* (x_0, y_0) *be a non-fixed point for* (2.7). *Then there exists a neighbourhood* $U(x_0, y_0)$ *such that the set* $\mathcal{O}(x_0, y_0) \cap U$, *that is the orbit of* (x_0, y_0) *in* U, *is the graph of a function.*
In particular, if $f(x_0, y_0) \neq 0$ *there exist* $\varepsilon > 0$ *and a* C^k *function* $h : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ *such that*

$$\mathcal{O}(x_0, y_0) \cap (x_0 - \varepsilon, x_0 + \varepsilon) = \{(x, h(x)) \,:\, x \in (x_0 - \varepsilon, x_0 + \varepsilon)\}\,,$$

and $h(x)$ *satisfies the Cauchy system*

$$\begin{cases} \frac{dy}{dx} = \frac{g(x,y)}{f(x,y)} \\ y(x_0) = y_0 \\ x \in (x_0 - \varepsilon, x_0 + \varepsilon) \end{cases} \tag{2.9}$$

Instead, if $g(x_0, y_0) \neq 0$ *the analogous statement holds by interchanging the roles of* x *and* y *and of* f *and* g.

Proof. If $f(x_0, y_0) \neq 0$ there exists $\varepsilon > 0$ such that $f(x, y) \neq 0$ for all $(x, y) \in B_{2\varepsilon}(x_0, y_0)$. Let $h(x)$ be a solution to system (2.9) and define the C^k function $I(x, y) = y - h(x)$ on $\{x \in (x_0 - \varepsilon, x_0 + \varepsilon)\} \cap B_{2\varepsilon}(x_0, y_0)$. Then with respect to system (2.7)

$$\begin{aligned} \dot{I}|_{\{I=0\}} &= \left(\dot{y} - h'(x)\,\dot{x}\right)|_{\{I=0\}} = \left(g(x, y) - h'(x)\,f(x, y)\right)|_{y=h(x)} = \\ &= g(x, h(x)) - h'(x)\,f(x, h(x)) \equiv 0. \end{aligned}$$

Therefore $I_0 := \{y = h(x)\}$ is an invariant set in a neighbourhood $U(x_0, y_0)$ containing (x_0, y_0). Then $I_0 = \mathcal{O}(x_0, y_0) \cap U$, and the proposition is proved.
The analogous argument works if $g(x_0, y_0) \neq 0$. ☐

The solutions to system (2.9) are called *isoclines* for (2.7).

Example 2.7 (Predator-prey Lotka-Volterra models). We apply the method of finding isoclines to prove the existence of periodic orbits in a predator-prey Lotka-Volterra system. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in a predator-prey relationship. The population x predates on the population y, hence the system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(-A + b_1\,y) \\ \dot{y} = y\,(B - b_2\,x) \end{cases} \tag{2.10}$$

with $A, B, b_1, b_2 > 0$. The system has two fixed points, $P_0 = (0,0)$ and $P_1 = (B/b_2\,,\, A/b_1)$. The point P_0 is hyperbolic and it is a saddle with stable and unstable manifolds given by the x and y axis respectively, whereas the point P_1 is not hyperbolic being a center.

Let us find the isoclines of (2.10). When $x_0 \neq 0$ and $y_0 \neq A/b_1$ we can write

$$\begin{cases} \frac{dy}{dx} = \frac{y\,(B-b_2\,x)}{x\,(-A+b_1\,y)} \\ y(x_0) = y_0 \end{cases}$$

which has a local solution given implicitly by the equality

$$\int_{y_0}^{y} \frac{-A + b_1\,y}{y}\,ds = \int_{x_0}^{x} \frac{B - b_2\,x}{x}\,dt \quad \Leftrightarrow \quad I(x,y) = I(x_0,y_0)$$

where

$$I(x,y) := A\,\log y + B\,\log x - b_1\,y - b_2\,x\,.$$

We have thus found that $I(x,y)$ is a first integral for (2.10), hence the orbits lie on its level sets. Then, it is immediate to find that $I(x,y)$ has a point of global minimum at P_1, therefore the levels sets $\{I(x,y) = c\}$ are closed curves for c bigger than $I(P_1)$ but sufficiently close to it. Hence, the orbits on these level sets are periodic[3].

The field and the symmetries of the system

Here we introduce two ideas to draw the phase portrait of a system. Both ideas work in all dimensions but are particularly simple to apply in the two dimensional case.

The first idea uses the property of the field to be tangent to the orbits of a system. Therefore, in principle, one can obtain the orbits of a system simply by drawing the field in all the points of the phase space. In practice,

[3]It can be proved that all level sets are closed curves, therefore all orbits different from the axes and the fixed points are periodic.

it is useful to draw the behaviour of the field on some curves. For example, it is a good idea to draw the lines on which the single components of the field vanish (the intersection of these lines give the fixed points) and to obtain the direction of the field in all the regions of the phase space between these lines.

A more theoretical idea to apply is to look for symmetries of the system. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$, the associated system $\dot{\underline{x}} = F(\underline{x})$, and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we give the following definition.

Definition 2.10. Given a vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ and a diffeomorphism $S : \mathbb{R}^n \to \mathbb{R}^n$, we say that the system $\dot{\underline{x}} = F(\underline{x})$ is *symmetric* with respect to S if

$$d_{\underline{x}} S(F(\underline{x})) = \pm F(S(\underline{x})) \,, \quad \forall\, \underline{x} \in \mathbb{R}^n \,.$$

A simple case is the case of systems symmetric with respect to linear transformations. That is there exists an invertible matrix $S \in M(n \times n, \mathbb{R})$ such that $S\, F(\underline{x}) = \pm F(S\, \underline{x})$.

Proposition 2.22. *If the system* (2.7) *in* $\mathbb{R}^2$ *is symmetric with respect to a diffeomorphism* S*, given a trajectory* $(x(t), y(t))$ *of the system, the curve* $(\tilde{x}(t), \tilde{y}(t))$ *defined as*

$$(\tilde{x}(t), \tilde{y}(t)) = \begin{cases} S(x(t), y(t)) \,, & \text{if } d_{\underline{x}} S(F(\underline{x})) = F(S(\underline{x})), \\ S(x(-t), y(-t)) \,, & \text{if } d_{\underline{x}} S(F(\underline{x})) = -F(S(\underline{x})), \end{cases}$$

is a solution to (2.7)*.*

Proof. It is enough to compute $(\dot{\tilde{x}}(t), \dot{\tilde{y}}(t))$. □

Example 2.8. We show how the proposition works in two easy cases. Let us consider the system (2.7) with the assumption that $f(-x, -y) = -f(x, y)$ and $g(-x, -y) = -g(x, y)$. The field $F(x, y) = (f(x, y), g(x, y))$ satisfies $F(-x, -y) = -F(x, y)$, hence it is symmetric with respect to the linear transformation $S(x, y) = (-x, -y)$ and

$$d_{(x,y)} S(F(x, y)) = -F(x, y) = F(S(x, y)).$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (-x(t), -y(t))$. Indeed, we have

$$\dot{\tilde{x}}(t) = -\dot{x}(t) = -f(x(t), y(t)) = f(-x(t), -y(t)) = f(\tilde{x}(t), \tilde{y}(t)) \,,$$

$$\dot{\tilde{y}}(t) = -\dot{y}(t) = -g(x(t), y(t)) = g(-x(t), -y(t)) = g(\tilde{x}(t), \tilde{y}(t)) \,.$$

The other case considered in the proposition is obtained for Hamiltonian systems in $\mathbb{R}^2$ with Hamiltonian function of the form (2.2). In this case the field is $F(x,y) = (y, -W'(x))$ and the system is symmetric with respect to the linear transformation $S(x,y) = (x, -y)$ since

$$d_{(x,y)}S(F(x,y)) = (y, W'(x)) = -F(x,-y) = -F(S(x,y)) .$$

Then, given a trajectory $(x(t), y(t))$ of the system, we show that another trajectory is given by $(\tilde{x}(t), \tilde{y}(t)) = (x(-t), -y(-t))$. Indeed,

$$\begin{aligned} \dot{\tilde{x}}(t) &= -\dot{x}(-t) = -y(-t) = \tilde{y}(t) , \\ \dot{\tilde{y}}(t) &= \dot{y}(-t) = -W'(x(-t)) = -W'(\tilde{x}(t)) . \end{aligned}$$

Periodic orbits: non-existence

We describe two methods to prove non-existence of periodic orbits in a region of the phase space. The first is of pure topological nature and the second uses the analytical nature of the differential equation (2.7).

Definition 2.11. Let $\Gamma \subset \mathbb{R}^2$ be a simple closed curve. Given a vector field $F(x,y) = (f(x,y), g(x,y))$ without fixed points on Γ, the *Poincaré index of* Γ, denoted by $I_F(\Gamma)$, is the number of turns that F makes counterclockwise as a point goes round Γ. It can be computed as

$$I_F(\Gamma) := \frac{1}{2\pi} \int_\Gamma d\left(\arctan \frac{g}{f}\right) = \frac{1}{2\pi} \int_\Gamma \frac{f\,dg - g\,df}{f^2 + g^2}$$

Proposition 2.23. *Given a vector field F on $\mathbb{R}^2$, the Poincaré index of a curve has the following properties:*

(i) let $t \mapsto \Gamma_t$ be a continuous family of simple closed curves, then $I_F(\Gamma_t)$ is constant as long as no Γ_t contains a fixed point of F;

(ii) let Γ be a simple closed curve not containing fixed points of F which can be written as $\Gamma = \Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two simple closed curves not containing fixed points of F. Then $I_F(\Gamma) = I_F(\Gamma_1) + I_F(\Gamma_2)$;

(iii) if Γ is a periodic orbit then $I_F(\Gamma) = +1$.

Definition 2.12. Let (x_0, y_0) be an isolated fixed point of a vector field F. The *Poincaré index of* (x_0, y_0), $I_F(x_0, y_0)$, is the Poincaré index of any simple closed curve Γ encircling (x_0, y_0) and no other fixed point of F.

Proposition 2.24. *Let (x_0, y_0) be a fixed point of a C^1 vector field F on $\mathbb{R}^2$ with $\det(JF(x_0, y_0)) \neq 0$. Then:*

(i) if (x_0, y_0) is a node, a star, an improper node, a focus or a centre, then $I_F(x_0, y_0) = +1$;

(ii) if (x_0, y_0) is a saddle, then $I_F(x_0, y_0) = -1$.

Putting together Propositions 2.23 and 2.24, we obtain information on regions of a phase space where a periodic orbit may exist or not. For example, it may not exist a periodic orbit encircling only a saddle. Each periodic orbit has to encircle sets of isolated fixed points for which the sum of their Poincaré indices is +1.

Example 2.9. Let us consider the system

$$\begin{cases} \dot{x} = x \\ \dot{y} = y^2 \end{cases}$$

then $I_F(0, 0) = 0$.

Example 2.10. Let us consider the system

$$\begin{cases} \dot{x} = x^2 - y^2 \\ \dot{y} = 2xy \end{cases}$$

then $I_F(0, 0) = 2$.

Proposition 2.25 (Curl method)**.** *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that the vector field $F(x, y) = (f(x, y), g(x, y))$ satisfies*

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial g}{\partial x}(x, y)\,, \quad \forall\,(x, y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T parametrised by the solution $\gamma(t)$ of the Cauchy problem

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y) \\ (x(0), y(0)) = \gamma(0) \end{cases}$$

Then $\gamma(T) = \gamma(0)$ and $\gamma'(t) = F(\gamma(t))$ for all $t \in \mathbb{R}$.

By assumption and Poincaré's lemma, the vector field F is conservative in U, that is there exists a C^1 function $h : U \to \mathbb{R}$ such that $F = \nabla h$. Then

$$0 = h(\gamma(T)) - h(\gamma(0)) = \int_0^T \frac{d}{dt}(h \circ \gamma)(t)\, dt = \int_0^T \big\langle \nabla h(\gamma(t)), \gamma'(t) \big\rangle\, dt =$$

$$= \int_0^T \langle F(\gamma(t)), F(\gamma(t)) \rangle\, dt = \int_0^T \|F(\gamma(t))\|^2\, dt$$

which is a contradiction because $\|F(\gamma(t))\| \neq 0$ for all t. □

Remark 2.26. The curl method can be easily extended to a differential equation in $\mathbb{R}^n$ with vector field F. By repeating the last part of the proof of Proposition 2.25 one can show that

Proposition 2.27 (Gradient systems). *If there exists $h : \mathbb{R}^n \to \mathbb{R}$ such that $F = \nabla h$, then there are no periodic orbits for the differential equations $\dot{\underline{x}} = F(\underline{x})$.*

Proposition 2.28 (Bendixson-Dulac method). *Let $U \subset \mathbb{R}^2$ be a simply connected open set and assume that there exists a C^1 function $\rho : U \to \mathbb{R}$ such that for the vector field $F(x,y) = (f(x,y), g(x,y))$ it holds*

$$\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) > 0 \ (or < 0), \quad \forall\, (x,y) \in U$$

Then in U there exist no periodic orbits for the vector field F.

Proof. Let $\Gamma \subset U$ be a periodic orbit of period T and let A be the region enclosed by Γ. Then applying Gauss-Green Theorem

$$0 < \iint_A \left(\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) \right) dxdy = \int_\Gamma (-\rho\, g\, dx + \rho\, f\, dy) =$$

$$= \int_0^T \rho(x(t), y(t))\, \big(-g(x(t), y(t))\, \dot{x}(t) + f(x(t), y(t))\, \dot{y}(t)\big) = 0$$

where we have used that $\Gamma = (x(t), y(t))$ for $t \in [0,T]$ and $(x(t), y(t))$ is a solution of the differential equation associated to the vector field F. □

Example 2.11 (Species in competition). In Example 2.7 we have shown that predator-prey Lotka-Volterra models admit periodic orbits. Now we show that there are no periodic orbits in a Lotka-Volterra model for species in competition. Let $x, y \in \mathbb{R}_0^+$ denote the population of two species in competition for the same resources on a finite environment. The system of differential equations for x and y is of the form

$$\begin{cases} \dot{x} = x\,(A - a_1\,x - b_1\,y) = f(x,y) \\ \dot{y} = y\,(B - b_2\,x - a_2\,y) = g(x,y) \end{cases} \tag{2.11}$$

with $A, B, a_1, a_2, b_1, b_2 > 0$.

The axes are invariant sets, hence the simply connected set

$$U = \big\{(x,y) \in \mathbb{R}^2 \,:\, x > 0,\, y > 0\big\}$$

is also invariant. Consider the C^1 function $\rho(x,y) = 1/(xy)$ on U. We have

$$\frac{\partial(\rho \cdot f)}{\partial x}(x,y) + \frac{\partial(\rho \cdot g)}{\partial y}(x,y) = -\frac{a_1}{y} - \frac{a_2}{x} < 0\,, \quad \forall\,(x,y) \in U.$$

Hence, by Proposition 2.28, there are no periodic orbits in U.

Remark 2.29. The Bendixson-Dulac method uses the divergence of a vector field. For differential equations in $\mathbb{R}^n$, $n \geq 3$, it gives different information.

Proposition 2.30. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 vector field such that there exists a constant $k > 0$ for which $div(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$. Then the flow associated to F contracts the volumes.*

Proof. For $A \subset \mathbb{R}^n$ let $\phi_t(A)$ be the evolution of the set at time t, and let m be the n-dimensional Lebesgue measure. By applying the same ideas in the proof of Liouville Theorem 2.15, we obtain

$$\mathrm{vol}(\phi_t(A)) = \int_A \exp\left(\int_0^t \mathrm{div}(F)(\phi_s(\underline{x}))\,ds\right)\,dm.$$

If $\mathrm{div}(F)(\underline{x}) \leq -k$ for all $\underline{x} \in \mathbb{R}^n$ then

$$\mathrm{vol}(\phi_t(A)) \leq e^{-kt}\,\mathrm{vol}(A)\,, \quad \forall\, t \geq 0,$$

and the proof is finished. □

Periodic orbits: existence in general

Theorem 2.31 (Poincaré - Bendixson). *Let F be a C^1 vector field in $\mathbb{R}^2$, and assume that there exists a non-empty region $D \subset \mathbb{R}^2$ which is compact and does not contain fixed points of F. If for some $\underline{x}_0$ there exists t_0 such that $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$, then there exists a periodic orbit $\Gamma \subset D$ and $\Gamma = \omega(\underline{x}_0)$.*

For the proof we need some preliminaries. Given the differential equation $\dot{\underline{x}} = F(\underline{x})$ in $\mathbb{R}^2$ with $F \in C^1$ and any non-fixed point $\underline{y}$ of F, we call *transversal line at* $\underline{y}$ the line $\ell(\underline{y})$ which is the image of the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $\gamma(u) = \underline{y} + u\underline{v}$, where $\underline{v}$ is a vector applied at $\underline{y}$ which satisfies $\langle \underline{v}, F(\underline{y}) \rangle = 0$.

Definition 2.13. Given a non-fixed point $\underline{y}$ of F and a constant $k \in [0,1)$, we call *k-wide local section at $\underline{y}$* the set $S_k(\underline{y})$ obtained by taking the connected component containing $\underline{y}$ of the set of points $\underline{z} \in \ell(\underline{y})$ for which $|\sin(\widehat{\underline{v}F(\underline{z})})| > k$.

The k-wide local section at $\underline{y}$ is non-empty since $\underline{y} \in S_k(\underline{y})$, and there exists $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon) \subseteq S_k(\underline{y})$.

Proposition 2.32 (Local rectifiability of a vector field)**.** *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, there exists a diffeomorphisms $\psi : U(\underline{0}) \to V(\underline{y})$ which maps horizontal lines into the orbits of $\dot{\underline{x}} = F(\underline{x})$ passing through $S_k(\underline{y})$. That is $\psi(s,u) = \phi_s(\gamma(u))$ for all $(s,u) \in U(\underline{0})$.*

Applying Proposition 2.32, let $\sigma > 0$ and $N_\sigma := \{(s,u) \in U(\underline{0}) : |s| < \sigma\}$. Then we call *$\sigma$-rectangle of flux in $\underline{y}$* the set $\mathcal{N}_\sigma := \psi(N_\sigma)$. Then for each $\underline{z} \in \mathcal{N}_\sigma$ there exists a unique $s \in (-\sigma, \sigma)$ such that $\phi_s(\underline{z}) \in S_k(\underline{y})$.

Proposition 2.33. *Given a C^1 vector field F in $\mathbb{R}^2$, a non-fixed point $\underline{y}$ of F, and a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, let $\underline{z}$ be a point such that $\underline{y} = \phi_{t_0}(\underline{z})$ for some t_0. Then there exist $\varepsilon > 0$ and a continuous function $\tau : B_\varepsilon(\underline{z}) \to \mathbb{R}$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$.*

Proof. Let us define the function $p : \mathbb{R}^2 \to \mathbb{R}$ by $p(\underline{x}) = \langle \underline{x}, F(\underline{y}) \rangle$. We notice that $p(\underline{x}) = p(\underline{y})$ if and only if $\underline{x} \in \ell(\underline{y})$, in fact if $\underline{x} = \underline{y} + \underline{w}$ then

$$p(\underline{x}) = p(\underline{y}) + p(\underline{w}) = p(\underline{y}) \quad \Leftrightarrow \quad \langle \underline{w}, F(\underline{y}) \rangle = 0$$

Let then consider the regular function $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ given by $G(\underline{x}, t) = p(\phi_t(\underline{x}))$. Then by definition $G(\underline{z}, t_0) = p(\phi_{t_0}(\underline{z})) = p(\underline{y})$ and

$$\frac{\partial G}{\partial t}(\underline{z}, t_0) = p(\dot{\phi}_t(\underline{x}))\Big|_{\underline{x}=\underline{z}, t=t_0} = p(F(\phi_{t_0}(\underline{z}))) = p(F(\underline{y})) = \|F(\underline{y})\|^2 \neq 0$$

Hence we can apply the Implicit Function Theorem to G at $(\underline{z}, t_0)$ and prove the existence of $\varepsilon > 0$ and $\delta > 0$, and of a continuous function $\tau : B_\varepsilon(\underline{z}) \to (t_0 - \delta, t_0 + \delta)$ such that

$$p(\underline{y}) = G(\underline{x}, \tau(\underline{x})) = p(\phi_{\tau(\underline{x})}(\underline{x})) \qquad \forall\, \underline{x} \in B_\varepsilon(\underline{z})$$

It follows that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{z})$. □

We are now ready to prove Poincaré-Bendixson Theorem.

Proof of Theorem 2.31. Choose $\underline{x}_0 \in \mathbb{R}^2$ such that there exists t_0 for which $\phi_t(\underline{x}_0) \in D$ for all $t \geq t_0$. By Proposition 1.1, the set $\omega(\underline{x}_0) \subset D$ is non-empty, compact, and invariant. For any $\underline{x} \in \omega(\underline{x}_0)$ we show that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ, and that $\Gamma = \omega(\underline{x}_0)$.

Fix $\underline{x} \in \omega(\underline{x}_0)$, and let $\underline{y} \in \omega(\underline{x}) \subset \omega(\underline{x}_0)$, which is not a fixed point by assumption. Consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$.

Lemma 2.34. *The forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once.*

Proof. Since $\underline{y} \in \omega(\underline{x})$, there exists a point of $\mathcal{O}^+(\underline{x})$ in $\mathcal{N}_\sigma$, hence $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) \neq \emptyset$. Let's assume by contradiction that there exist $\underline{x}_1, \underline{x}_2 \in \mathcal{O}^+(\underline{x}) \cap S_k(\underline{y})$ with $\underline{x}_1 \neq \underline{x}_2$. Since $\underline{x} \in \omega(\underline{x}_0)$, also $\underline{x}_1, \underline{x}_2 \in \omega(\underline{x}_0)$ by invariance of the omega limit. Hence if $\mathcal{N}_\sigma(\underline{x}_1)$ and $\mathcal{N}_\sigma(\underline{x}_2)$ are disjoint σ-rectangles of flux, the forward orbit of $\underline{x}_0$ has countable points both in $\mathcal{N}_\sigma(\underline{x}_1)$ and in $\mathcal{N}_\sigma(\underline{x}_2)$. By the properties of the rectangles of flux, this implies that $\mathcal{O}^+(\underline{x}_0)$ intersects $S_k(\underline{y})$ countable many times, alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. We now show that this is not possible.

Let us denote by $\{\underline{z}_1, \underline{z}_2, \dots\}$ the points in $\mathcal{O}^+(\underline{x}_0) \cap S_k(\underline{y})$ cronologically ordered, that is $\underline{z}_1 = \phi_{t_1}(\underline{x}_0)$, $\underline{z}_2 = \phi_{t_2}(\underline{x}_0)$, and so on, with $t_1 < t_2 < \dots$. Given three points $\underline{z}_{n-1}, \underline{z}_n, \underline{z}_{n+1}$ and an ordering on $S_k(\underline{y})$ it must hold $\underline{z}_{n-1} < \underline{z}_n < \underline{z}_{n+1}$ or $\underline{z}_{n+1} < \underline{z}_n < \underline{z}_{n-1}$. Indeed let Σ denotes the Jordan curve given by the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ and the orbit $\cup_{t_{n-1} \leq t \leq t_n} \phi_t(\underline{x}_0)$, and let R be the region bounded by Σ. Then $\phi_t(\underline{x}_0) \in R$ for all $t > t_n$, because it cannot intersect any part of ∂R. It cannot intersect the orbit $\cup_{t_{n-1} \leq t \leq t_n} \phi_t(\underline{x}_0)$ by the uniqueness of solutions of a differential equation, and it cannot intersect the segment $\overline{\underline{z}_{n-1}\underline{z}_n}$ which is in $S_k(\underline{y})$, because the vector field points in the same direction in all the points of a local section. It follows that $\underline{z}_{n+1} \in R$ and it lies on the other side of $\underline{z}_{n-1}$ with respect to $\underline{z}_n$. It follows that the countable intersections of $\mathcal{O}^+(\underline{x}_0)$ with $S_k(\underline{y})$ must be ordered, so cannot be alternatively close to $\underline{x}_1$ and to $\underline{x}_2$. This shows that the forward orbit of $\underline{x}$ intersects $S_k(\underline{y})$ exactly once. □

We have thus proved that $\mathcal{O}^+(\underline{x}) \cap S_k(\underline{y}) = \{\phi_{\bar{t}}(\underline{x})\}$. Since $\underline{y} \in \omega(\underline{x})$ there is a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}) \to \underline{y}$, hence for m big enough $\phi_{t_m}(\underline{x}) \in \mathcal{N}_\sigma$. It follows that for m big enough, there exist $\tau_m \in \mathbb{R}$ such that $\phi_{t_m+\tau_m}(\underline{x}) \in S_k(\underline{y})$ for all m, hence $\phi_{t_m+\tau_m}(\underline{x}) = \phi_{\bar{t}}(\underline{x})$ for all m. It follows that there exists $T > 0$ such that $\phi_T(\underline{x}) = \underline{x}$. We have thus proved that $\mathcal{O}(\underline{x})$ is a periodic orbit Γ.

It remains to show that $\Gamma = \omega(\underline{x}_0)$. By invariance of the omega limit $\Gamma \subset \omega(\underline{x}_0)$. Let now $\underline{y} \in \Gamma$ and consider a k-wide local section at $\underline{y}$, $S_k(\underline{y})$, and a σ-rectangle of flux $\mathcal{N}_\sigma$ in $\underline{y}$. As discussed above, there exists a sequence $\{t_m\}$ such that $\phi_{t_m}(\underline{x}_0) \to \underline{y}$ and $\phi_{t_m}(\underline{x}_0) \in S_k(\underline{y})$, with $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$ for all m. Since $\phi_T(\underline{y}) = \underline{y}$, we can apply Proposition 2.33 and find $\varepsilon > 0$, $\delta > 0$, and a continuous function $\tau : B_\varepsilon(\underline{y}) \to (T-\delta, T+\delta)$ such that $\phi_{\tau(\underline{x})}(\underline{x}) \in S_k(\underline{y})$ for all $\underline{x} \in B_\varepsilon(\underline{y})$, and $\tau(\underline{y}) = T$. Hence, choosing $\tilde{\varepsilon} < \varepsilon$ if necessary, we have $\phi_T(\underline{x}) \in \mathcal{N}_\sigma$ for all $\underline{x} \in B_{\tilde{\varepsilon}}(\underline{y})$. Since for m big enough $\phi_{t_m}(\underline{x}_0) \in B_{\tilde{\varepsilon}}(\underline{y})$, it follows that $\phi_T(\phi_{t_m}(\underline{x}_0)) = \phi_{T+t_m}(\underline{x}_0) \in \mathcal{N}_\sigma$, and there exists $s_m \in (-\sigma, \sigma)$ such that $\phi_{T+t_m+s_m}(\underline{x}_0) \in S_k(\underline{y})$. Since $\phi_t(\underline{x}_0) \notin S_k(\underline{y})$ for $t \in (t_m, t_{m+1})$, it must hold $t_{m+1} = T+t_m+s_m$, hence $t_{m+1}-t_m \le T+\sigma$ for all m big enough.

We now consider a fixed $\eta > 0$. By continuity of the flux ϕ_t, there exists $\delta > 0$ such that if $d(\underline{z}_1, \underline{z}_2) < \delta$ then $d(\phi_t(\underline{z}_1), \phi_t(\underline{z}_2)) < \eta$ for all $t \in (-T-\sigma, T+\sigma)$. Hence, for m big enough such that $d(\phi_{t_m}(\underline{x}_0), \underline{y}) < \delta$, we have

$$d\Big(\phi_t(\phi_{t_m}(\underline{x}_0)), \phi_t(\underline{y})\Big) < \eta \qquad \forall\, t \in (-T-\sigma, T+\sigma)$$

Since $\underline{y} \in \Gamma$, so that $\mathcal{O}(\underline{y}) = \Gamma$, and $t_{m+1} - t_m \le T+\sigma$ for all m big enough, we have that

$$d\Big(\phi_t(\underline{x}_0), \Gamma\Big) < \eta \qquad \forall\, t \in (t_m, t_{m+1})$$

for m big enough. We can conclude that $d(\phi_t(\underline{x}_0), \Gamma) \to 0$ as $t \to +\infty$. Hence $\omega(\underline{x}_0) \subset \Gamma$. This shows that $\Gamma = \omega(\underline{x}_0)$, and concludes the proof of the theorem. □

Example 2.12. Let us consider the following system in polar coordinates

$$\begin{cases} \dot{\rho} = \rho(1-\rho^2) + \varepsilon\, f(\rho, \theta) \\ \dot{\theta} = 1 + \varepsilon\, g(\rho, \theta) \end{cases}$$

with $f, g \in C^1(\mathbb{R}^2)$. For $\varepsilon = 0$ the system admits the orbitally asymptotically stable periodic orbit $\Gamma = \{\rho = 1\}$. We now show that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a periodic orbit Γ_ε. Let

$$L = \max_{\rho \le 5}\, (|f| + |g|)$$

and $\varepsilon_0 = \frac{1}{4L}$. We now prove that if $\varepsilon < \varepsilon_0$ the set $D = \left\{\frac{1}{2} \le \rho \le 2\right\}$ satisfies the assumptions of Poincaré-Bendixson Theorem 2.31.

First of all for all $(\rho, \theta) \in D$

$$1 + \varepsilon\, g(\rho, \theta) \geq 1 - \varepsilon L > 1 - \varepsilon_0 L = \frac{3}{4}$$

so that D contains no fixed points of the system. Moreover

$$\dot{\rho}|_{\rho=2} = -6 + \varepsilon\, f(2, \theta) < -6 + \varepsilon\, L < -6 + \varepsilon_0\, L = -6 + \frac{1}{4} < 0$$

and

$$\dot{\rho}|_{\rho=\frac{1}{2}} = \frac{3}{8} + \varepsilon\, f\left(\frac{1}{2}, \theta\right) > \frac{3}{8} - \varepsilon\, L > \frac{3}{8} - \varepsilon_0\, L = \frac{1}{8} > 0$$

so that on ∂D the vector field is always directed towards the inside of D. This implies that for all $\underline{x} \in \partial D$ and for all $t > 0$ it holds $\phi_t(\underline{x}) \in D$, and completes the proof.

Finally, we state a result which extends Theorem 2.31 to the case of regions with fixed points.

Theorem 2.35. *Let F be a C^1 vector field in $\mathbb{R}^2$, and let $D \subset \mathbb{R}^2$ be a non-empty bounded positively invariant region containing at most a finite number of fixed points for F. Then, for all $\underline{x} \in D$, the set $\omega(\underline{x})$ is non-empty and one of the following possibilities holds:*

- *$\omega(\underline{x})$ is a fixed point;*
- *$\omega(\underline{x})$ is a periodic orbit;*
- *$\omega(\underline{x})$ consists of a finite number of fixed points and heteroclinic orbits connecting them.*

2.6 Exercises

2.1. Draw the phase portrait of the linear system $\dot{\underline{x}} = A\,\underline{x}$ in $\mathbb{R}^2$ and find the stable, unstable, and central eigenspace of $\underline{0}$, with A given by:

(a) $A = \begin{pmatrix} 5 & 4 \\ 2 & 7 \end{pmatrix}$ (b) $A = \begin{pmatrix} -8 & 0 \\ 1 & -6 \end{pmatrix}$ (c) $A = \begin{pmatrix} -8 & 6 \\ -9 & 13 \end{pmatrix}$

(d) $A = \begin{pmatrix} -8 & 4 \\ -1 & -4 \end{pmatrix}$ (e) $A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$ (f) $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

(g) $A = \begin{pmatrix} -7 & -5 \\ 1 & -5 \end{pmatrix}$ (h) $A = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$ (i) $A = \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}$

2.2. For the following systems, find the critical points and study their linear stability.

$$(a)\ \begin{cases} \dot{x} = -2x(x-1)(2x-1) \\ \dot{y} = -2y \end{cases} \qquad (b)\ \begin{cases} \dot{x} = x(4-2x-y) \\ \dot{y} = y(3-x-y) \end{cases}$$

$$(c)\ \begin{cases} \dot{x} = -y + x^3 \\ \dot{y} = x + y^3 \end{cases} \qquad (d)\ \begin{cases} \dot{x} = e^{(x+y)} + y \\ \dot{y} = y - xy \end{cases}$$

$$(e)\ \begin{cases} \dot{x} = 2xy \\ \dot{y} = y^2 - x^2 \end{cases} \qquad (f)\ \begin{cases} \dot{x} = x(60-4x-3y) \\ \dot{y} = y(42-3x-2y) \end{cases}$$

2.3. Find a Lyapunov function to study the stability of the fixed point $(0,0)$ for the following systems:

$$(a)\ \begin{cases} \dot{x} = y - 3x^3 \\ \dot{y} = -x - 7y^3 \end{cases} \qquad (b)\ \begin{cases} \dot{x} = -xy^4 \\ \dot{y} = yx^4 \end{cases}$$

$$(c)\ \begin{cases} \dot{x} = x - xy^4 \\ \dot{y} = y - y^3x^2 \end{cases} \qquad (d)\ \begin{cases} \dot{x} = x^2 - xy - x \\ \dot{y} = y^2 + 2xy - 7y \end{cases}$$

2.4. Determine the stability of the fixed point $(0,0)$ varying $\mu \in \mathbb{R}$ for the system

$$\begin{cases} \dot{x} = (\mu x + 2y)(z+1) \\ \dot{y} = (-x + \mu y)(z+1) \\ \dot{z} = -z^3 \end{cases}$$

2.5. Find the fixed points and study their stability varying $\mu \in \mathbb{R}$, $\mu \neq 4$, for the system

$$\begin{cases} \dot{x} = \mu\, x^3 - x^5 \\ \dot{y} = (2\mu y + z)(x-2) \\ \dot{z} = (-2y + \mu z)(x-2) \end{cases}$$

2.6. Draw the phase portrait for a mechanical Hamiltonian system with $H(x,y)$ of the form (2.4) with $m = 1$ and potential energy W given by:

(a) $W(x) = \frac{1}{3}x^2 + \frac{1}{9}x^3 - \frac{1}{4}x^4$;

(b) $W(x) = x \log(1 + x^2)$;

(c) $W(x) = \begin{cases} e^{-x^2}, & x \leq 0 \\ \cos(\sqrt{2}\,x), & x \geq 0 \end{cases}$;

(d) $W(x) = -\frac{\sin x}{x}$.

2.7. Consider the system

$$\begin{cases} \dot{x} = \frac{1}{2}y \\ \dot{y} = -(1+\mu)x + \mu x^2 + x^3 \end{cases}$$

varying $\mu \in \mathbb{R}$. Show that it is a mechanical Hamiltonian system writing down the Hamiltonian function. Let denote by $(x_\mu(t,0), y_\mu(t,y_0))$ the solution to the system with initial condition $(x(0), y(0)) = (0, y_0)$, then find

$$y^*(\mu) := \inf\{y_0 > 0 \,:\, \lim_{t \to +\infty} x_\mu(t) = +\infty\}\,.$$

2.8. Draw the phase portrait for the following systems:

$$(a) \begin{cases} \dot{x} = y - x^2 \\ \dot{y} = x - 2 \end{cases} \qquad (b) \begin{cases} \dot{x} = \sin x\,(-0.1\cos x - \cos y) \\ \dot{y} = \sin y\,(\cos x - 0.1\cos y) \end{cases} \text{ on } [0,\pi]^2$$

$$(c)\ \begin{cases} \dot{x} = x^2 - 1 \\ \dot{y} = -xy + x^2 - 1 \end{cases} \qquad (d)\ \begin{cases} \dot{x} = y\ \cos x \\ \dot{y} = \sin x \end{cases}$$

$$(e)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x \end{cases} \qquad (f)\ \begin{cases} \dot{x} = y \\ \dot{y} = x^3 - x + \frac{1}{2}y \end{cases}$$

2.9. For the following systems, study the existence of a periodic orbit entirely contained in $\{x^2 + y^2 \geq 2\}$:

$$(a)\ \begin{cases} \dot{x} = x^3 - x + y^2 \\ \dot{y} = -2y \end{cases} \qquad (b)\ \begin{cases} \dot{x} = \frac{x^3}{1+x^4+y^4} \\ \dot{y} = \frac{y^3}{1+x^4+y^4} \end{cases}$$

2.10. Study the existence of a periodic orbit for the system

$$\begin{cases} \dot{x} = x\ \sqrt{x^2 + y^2} - 3x\,(x^2 + y^2) + \frac{1}{10}y^5 \\ \dot{y} = y\ \sqrt{x^2 + y^2} - 3y\,(x^2 + y^2) - \frac{1}{10}x^5 \end{cases}$$

Chapter 3

Discrete-time dynamical systems

In this chapter we consider discrete-time dynamical systems as defined in Definition 1.2. Hence we need to specify a set X and a map $T : X \to X$. The properties of X and T may vary and give rise to different areas of research. Here we assume that X is a locally compact connected metric space and T is a continuous map, and call (X, T) a *discrete-time continuous dynamical system*. In many situations one can simply consider X to be an interval of the real line, and in fact some results of this chapter hold only for one-dimensional spaces X or even for compact intervals of the real line.

We start with simple definitions.

Definition 3.1. Let (X, T) and $(\tilde{X}, \tilde{T})$ be two discrete-time continuous dynamical systems. We say that $(\tilde{X}, \tilde{T})$ is a *topological factor* of (X, T) if there exists a continuous map $h : X \to \tilde{X}$ that is surjective and satisfies

$$\tilde{T} \circ h = h \circ T\,. \tag{3.1}$$

If the map $h : X \to \tilde{X}$ is a homeomorphism and satisfies (3.1) then we say that (X, T) and $(\tilde{X}, \tilde{T})$ are *topologically conjugate* and h is a *topological conjugacy.*

Example 3.1. Let's consider the full shift $(\Omega_{\mathcal{A}}, \mathbb{N}_0, \sigma)$ on two symbols $\mathcal{A} = \{0, 1\}$ of Example 1.8, and the Bernoulli map T_2 on S^1 of Example 1.7. Let $J_0 = [0, 1/2)$ and $J_1 = [1/2, 1)$ be a partition of S^1, and let the map $h : \Omega_{\{0,1\}} \to S^1$ be defined by

$$\omega = (\omega_i)_{i\in\mathbb{N}_0} \mapsto h(\omega) = \bigcap_{i\in\mathbb{N}_0} T_2^{-i}(J_{\omega_i})\,.$$

The map h is continuous and surjective, and satisfies $T_2 \circ h = h \circ \sigma$. Then the Bernoulli map is a topological factor of the full shift on two symbols.

Example 3.2. Let's consider the Tent map T_s with $s = 2$ of Example 1.5, and the logistic map T_λ with $\lambda = 4$ of Example 1.6. Let the map $h : [0,1] \to [0,1]$ be defined by

$$[0,1] \ni x \mapsto h(x) = \sin^2\left(\frac{\pi}{2}x\right) .$$

The map h is a homeomorphism, and satisfies $T_4 \circ h = h \circ T_2$. Hence the Tent map T_s with $s = 2$ is topologicall conjugate to the logistic map T_λ with $\lambda = 4$.

Remark 3.1. In some situations it is interesting to study the regularity of a conjugacy. For example, if T and $\tilde{T}$ are C^k maps, with $k \in \mathbb{N}_0 \cup \{\infty, \omega\}$, a natural question is whether there exists a conjugacy h between the systems (X,T) and $(\tilde{X}, \tilde{T})$ which is of class C^k. If it exists we say that (X,T) and $(\tilde{X}, \tilde{T})$ are C^k *conjugate.*

3.1 Stability in one dimension

Let $T : X \to X$ be a continuous map of a one-dimensional space $X = [a,b], (a,b), [a,+\infty), (a,+\infty), (-\infty,b], (-\infty,b), \mathbb{R}, S^1$.

Definition 3.2. A fixed point $x_0 \in X$ of T is called *attractive* if there exists $\delta > 0$ such that, for all $x \in B_\delta(x_0)$, one has $T^n(x) \in B_\delta(x_0)$ for all $n \geq 0$, and $T^n(x) \to x_0$ as $n \to +\infty$.
A fixed point $x_0 \in X$ is called *repulsive* if there exists $\delta > 0$ such that, for all $x \in B_\delta(x_0)$, $x \neq x_0$, there exists $\bar{n} \in \mathbb{N}$ for which $T^{\bar{n}}(x) \notin B_\delta(x_0)$.

To study the dynamics in a neighbourhod of a fixed point x_0, first it is useful to try the linearization approach. Let T be differentiable at x_0. Then, there exists $\varepsilon > 0$ such that for all $x \in B_\varepsilon(x_0)$

$$T(x) = T(x_0) + T'(x_0)(x-x_0) + o(|x-x_0|) = x_0 + T'(x_0)(x-x_0) + o(|x-x_0|) .$$

Hence,

$$|T(x) - x_0| = |T'(x_0)|\,|x - x_0| + o(|x - x_0|) . \tag{3.2}$$

We deduce that, at the first order, it is the derivative $T'(x_0)$ which may determine whether the orbit of a point $x \in B_\varepsilon(x_0)$ gets closer or further from the fixed point x_0. This justifies the following definition.

Definition 3.3. Let T be differentiable at a fixed point x_0. The fixed point $x_0 \in X$ is called *hyperbolic* if $|T'(x_0)| \neq 1$.

Theorem 3.2. *Let x_0 be a hyperbolic fixed point for a map T which is differentiable at x_0. If $|T'(x_0)| < 1$ then the point is attractive, if $|T'(x_0)| > 1$ then the point is repulsive.*

Proof. Let $|T'(x_0)| < 1$ and fix $c \in (|T'(x_0)|, 1)$. If we choose $\delta > 0$ such that $|T'(x)| \leq c$ for all $x \in B_\delta(x_0)$, then we have that for all $n \geq 1$

$$|T^n(x) - x_0| \leq c^n \, |x - x_0| \, , \quad \forall \, x \in B_\delta(x_0) \, . \tag{3.3}$$

From (3.3) and $c \in (0, 1)$, it follows that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 0$ and $T^n(x) \to x_0$ as $n \to +\infty$.

We now prove (3.3) by induction. For $n = 1$, for all $x \in B_\delta(x_0)$ there exists ξ_1 between x and x_0 such that

$$|T(x) - x_0| = |T(x) - T(x_0)| = |T'(\xi_1)| \, |x - x_0| \leq c \, |x - x_0| \, ,$$

where $|T'(\xi_1)| \leq c$ since $\xi_1 \in B_\delta(x_0)$. Then, let's assume that (3.3) holds for a given n, and show that it holds for $n + 1$. There exists ξ_n between $T^n(x)$ and x_0 such that

$$|T^{n+1}(x) - x_0| = |T(T^n(x)) - T(x_0)| = |T(\xi_n)| \, |T^n(x) - x_0| \leq$$
$$\leq c \cdot c^n \, |x - x_0| = c^{n+1} \, |x - x_0| \, ,$$

since $\xi_n \in B_\delta(x_0)$.

Let now $|T'(x_0)| > 1$, and first consider the case $T'(x_0) > 1$. Then we fix $c \in (1, T'(x_0))$ and choose $\delta > 0$ such that $T'(x) \geq c$ for all $x \in B_\delta(x_0)$. We now argue by contradiction and assume that there exists $x \in B_\delta(x_0)$, $x \neq x_0$, such that $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$. Then, we can repeat the argument above to show that

$$|T^n(x) - x_0| \geq c^n \, |x - x_0| \, , \quad \forall \, n \geq 1 \, ,$$

from which we find that $|T^n(x) - x_0| \to +\infty$ as $n \to +\infty$ since $c > 1$. This gives the contradiction with the assumption $T^n(x) \in B_\delta(x_0)$ for all $n \geq 1$.

A similar argument works in the case $|T'(x_0)| > 1$ and $T'(x_0) < -1$. $\square$

When the fixed point is not hyperbolic, the approach in (3.2) suggests that the higher derivatives of T at x_0 may give some information.

Definition 3.4. A fixed point $x_0 \in X$ is called *semi-attractive from the left* if there exists $\delta > 0$ such that it is attractive for points on $(x_0 - \delta, x_0)$ and repulsive for points on $(x_0, x_0 + \delta)$. A fixed point $x_0 \in X$ is called *semi-attractive from the right* if there exists $\delta > 0$ such that it is attractive for points on $(x_0, x_0 + \delta)$ and repulsive for points on $(x_0 - \delta, x_0)$.

Proposition 3.3. *Let x_0 be a fixed point for a map T which is differentiable at x_0 with $|T'(x_0)| = 1$. The following possibilities hold:*

(i) *Let $T'(x_0) = 1$ and assume that $T \in C^2(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, and $T''(x_0) \neq 0$. Then,*

- *If $T''(x_0) > 0$, then x_0 is semi-attractive from the left;*
- *If $T''(x_0) < 0$, then x_0 is semi-attractive from the right;*

(ii) *Let $T'(x_0) = 1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$, that $T''(x_0) = 0$, and $T'''(x_0) \neq 0$. Then,*

- *If $T'''(x_0) > 0$, then x_0 is repulsive;*
- *If $T'''(x_0) < 0$, then x_0 is attractive;*

(iii) *Let $T'(x_0) = -1$ and assume that $T \in C^3(B_\varepsilon(x_0))$ for some $\varepsilon > 0$. Then we look at $ST(x_0)$, the Schwarzian derivative of T at x_0, where*

$$ST(x) := \frac{T'''(x)}{T'(x)} - \frac{3}{2}\left(\frac{T''(x)}{T'(x)}\right)^2 . \tag{3.4}$$

Then,

- *If $ST(x_0) > 0$, then x_0 is repulsive;*
- *If $ST(x_0) < 0$, then x_0 is attractive.*

Proof. The proofs of (i) and (ii) are immediate from the graphical approach. Let us prove (iii). Since $T'(x_0) = -1$, in a neighborhood of x_0 the map T is order-reversing. We look at $G := T^2$ for which $G(x_0) = x_0$, and use that x_0 has the same stability for G and T. We have

$$G'(x) = T'(T(x))\, T'(x) \quad \Rightarrow \quad G'(x_0) = (T'(x_0))^2 = 1\,,$$

$$G''(x) = T''(T(x))\,(T'(x))^2 + T'(T(x))\, T''(x)$$

$$\Rightarrow \quad G''(x_0) = T''(x_0)\Big((T'(x_0))^2 - T'(x_0)\Big) = 0\,.$$

Moreover $G \in C^3(B_\varepsilon(x_0))$, hence we can compute $G'''(x_0)$. It holds

$$G'''(x) = T'''(T(x))\,(T'(x))^3 + 3\,T''(T(x))\,T'(x)\,T''(x) + T'(T(x))\,T'''(x)$$

$$\Rightarrow \quad G'''(x_0) = T'''(x_0)\Big((T'(x_0))^3 + T'(x_0)\Big) + 3\,(T''(x_0))^2\,T'(x_0)$$

$$\Rightarrow \quad G'''(x_0) = 2\,ST(x_0)\,.$$

The result follows from (ii). □

We conclude this section by studying the stability for periodic orbits.

Definition 3.5. Let x_0 be a periodic point for T with minimal period p. The orbit $\mathcal{O}(x_0)$ is called *attractive* (respectively *repulsive*) if x_0 is an attractive (respectively repulsive) fixed point for T^p.

Remark 3.4. Let x_0 be a periodic point for T with minimal period p. If $T \in C^1$, it is a straightforward corollary of the chain rule that the derivative of T^p is the same on all the points of the orbit of x_0, i.e. $(T^p)'(T^i(x_0)) = (T^p)'(x_0)$ for all $i = 0, \ldots, p-1$, since

$$(T^p)'(x_0) = \prod_{j=0}^{n-1} T'(T^j(x_0))\,.$$

3.2 Existence of periodic orbits

In this section $[a, b]$ denotes a compact interval of the real line. Given a finite number of points $\{x_k\}_{k=0,\dots,n}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b\,,$$

we consider the partition $\mathcal{J}$ of $[a, b]$ into the closed intervals $J_k = [x_{i-1}, x_i]$, $k = 1, \dots, n$.

Definition 3.6. Given a partition $\mathcal{J} = \{J_\ell\}$ of $[a, b]$ and two not necessarily distinct sets J_k and J_h of the partition, we say that J_k *T-covers* J_h *m-times*, with $m \in \mathbb{N} \cup \{\infty\}$, if there exist m open sub-intervals $K_1, \dots, K_m$ of J_k such that $K_i \cap K_j = \emptyset$ for $i \neq j$, and $T(\overline{K_i}) = J_h$ for all $i = 1, \dots, m$.

Definition 3.7. Given a partition $\mathcal{J} = \{J_\ell\}_{\ell=1,\dots,n}$ of $[a, b]$, the *T-graph of* $\mathcal{J}$ is a graph with nodes given by the indices $\{1, \dots, n\}$, and such that there are m-arcs from a set J_k to a set J_h if J_k T-covers J_h m-times. An *admissible path of length* $s \in \mathbb{N}$ on the T-graph of $\mathcal{J}$ is a sequence $J_{p(1)}J_{p(2)} \dots J_{p(s)}$ with $p(j) \in \{1, \dots, n\}$ and such that there is at least one arc from $J_{p(j)}$ to $J_{p(j+1)}$ for all $j = 1, \dots, s-1$. An admissible path of length $s \in \mathbb{N}$ is called *closed* if $p(s) = p(1)$.

Lemma 3.5. *If $J_{p(1)}J_{p(2)} \dots J_{p(s)}J_{p(s+1)}$ is an admissible closed path on the T-graph of a partition $\mathcal{J}$ with $s \in \mathbb{N}_0$, then there exists a point $x \in J_{p(1)}$ which is periodic for T with period s and such that $T^j(x) \in J_{p(j+1)}$ for all $j = 0, \dots, s$.*

Proof. Let us fix $K_{s+1} = \mathring{J}_{p(s+1)}$. Since the path $J_{p(1)}J_{p(2)} \dots J_{p(s)}J_{p(s+1)}$ is admissible, there exists a family $K_j \subset J_{p(j)}$, $j = 1, \dots, s$, of open intervals such that $T(K_j) = K_{j+1}$. Hence there exists an interval $K_1 \subset J_{p(1)}$ such that $T^s(K_1) = K_{s+1} \supseteq K_1$. The fixed-point theorem implies that there exists $x \in \overline{K_1}$ such that $T^s(x) = x$, moreover by construction $T^j(x) \in \overline{K_{j+1}} \subseteq J_{p(j+1)}$ for all $j = 0, \dots, s$. □

Remark 3.6. It is important to notice that Lemma 3.5 does not prove the existence of a periodic point with minimal period s. That the period s is minimal may be obtained by looking at the path used in the proof of the lemma.

Proposition 3.7. *Let $T : [a, b] \to [a, b]$ be a continuous map for which there exists a periodic orbit of odd period $m > 1$. Then T admits periodic orbits of minimal period n for all $n > m$, for all even $n < m$, and for $n = 1$.*

Proof. Let's assume that m is the smallest odd number greater than 1 for which T has a periodic orbit of period m[1]. In particular, m is the minimal period of the orbit. Let us denote by $p_1, p_2, \dots, p_m$ the points of the periodic orbit ordered in $[a, b]$, so that $T(p_1) > p_1$ and $T(p_m) < p_m$. It follows that there exists $\bar{h}$ such that $T(p_{\bar{h}}) > p_{\bar{h}}$ and $T(p_k) < p_k$ for all $k = \bar{h}+1, \dots, m$. Finally let $\mathcal{J}$ be the partition given by the points a, b and the points of the periodic orbit $p_1, p_2, \dots, p_m$, and let $J_0 := [a, p_1]$, $J_m := [p_m, b]$, and $J_k := [p_k, p_{k+1}]$ for $k \in \mathcal{N} := \{1, \dots, m-1\}$. By construction and the fact that $m > 2$ we have that one of the inequalities $T(p_{\bar{h}+1}) \leq p_{\bar{h}}$ and $T(p_{\bar{h}}) \geq p_{\bar{h}+1}$ is strict, hence $J_{\bar{h}}$ T-covers itself at least once. By Lemma 3.5, this gives the result for $n = 1$.

We now proceed by proving intermediate statements.

Step 1. There exists an admissible path on the T-graph of the partition $\mathcal{J}$ from $J_{\bar{h}}$ to any set J_k of the partition with $k \in \mathcal{N}$.

Let us define by recurrence the following subsets of the nodes $\mathcal{N}$ of the T-graph. We put $N_1 := \{\bar{h}\}$,

$$N_2 := \{r \in \mathcal{N} \,:\, J_{\bar{h}} \ T\text{-covers } J_r\} ,$$

and for $i \geq 3$

$$N_i := \{r \in \mathcal{N} \,:\, \exists\, s \in N_{i-1} \text{ such that } J_s \ T\text{-covers } J_r\} .$$

Since $m > 2$, each J_s with $s \in \mathcal{N}$, T-covers at least one set J_r with $r \neq s$. Moreover the fact that $J_{\bar{h}}$ T-covers itself implies that $\bar{h} \in N_i$ for all $i \geq 1$, hence $\{N_i\}$ is a non-decreasing sequence of sets. We conclude that there exists ℓ such that $N_\ell = N_{\ell+1} = \mathcal{N}$, since $N_\ell \neq \mathcal{N}$ implies that m is not the minimal period of the periodic orbit. This finishes the proof of this step.

Step 2. There exists $k \in \mathcal{N}$ such that J_k T-covers $J_{\bar{h}}$.

We argue by contradiction. If the thesis of this step is false, all points p_j of the periodic orbit with $j \leq \bar{h}$ have distinct images in the set $\{p_{\bar{h}+1}, \dots, p_m\}$, and analogously all points p_j of the periodic orbit with $j \geq \bar{h}+1$ have distinct images in the set $\{p_1, \dots, p_{\bar{h}}\}$. Since m is odd we get the contradiction.

Step 3. The T-graph of the partition $\mathcal{J}$ contains a loop starting from $J_{\bar{h}}$ through all the sets J_k with $k \in \mathcal{N}$, and contains one single arc from a set J_k with $k \in \mathcal{N}$ to $J_{\bar{h}}$.

We first show that the shortest admissible path from $J_{\bar{h}}$ to itself is of length m. Let $J_{\bar{h}} J_{p(2)} \dots J_{p(s)} J_{\bar{h}}$ be such path with length $s+1 < m$, there are two

[1] If not, we prove the result for such smallest odd number greater than 1 and obtain the proposition.

cases. If s is odd, by Lemma 3.5 there exists $x \in J_{\bar{h}}$ such that $T^s(x) = x$, but $s < m-1$ and we have a contradiction by the choice of m. If s is even, we can consider the admissible path $J_{\bar{h}} J_{p(2)} \dots J_{p(s)} J_{\bar{h}} J_{\bar{h}}$ which is of length $s+2$ and gives, by Lemma 3.5, the existence of a periodic point of period $s+1 < m$. Again we have a contradiction by the choice of m.

Let $J_{\bar{h}} J_{p(2)} \dots J_{p(m-1)} J_{\bar{h}}$ be the shortest admissible path from $J_{\bar{h}}$ to itself. All J_k appear at most once in this path, indeed if one J_k appears twice, we can construct a shorter admissible path from $J_{\bar{h}}$ to itself. It follows that this path is actually a loop starting from $J_{\bar{h}}$ through all the sets J_k with $k \in \mathcal{N}$. The same argument shows that the T-graph of the partition $\mathcal{J}$ contains one single arc from a set J_k with $k \in \mathcal{N}$ to $J_{\bar{h}}$.

Let us now relabel the sets of the partition $\mathcal{J}$ by letting $I_1 := J_{\bar{h}}$ and $I_2, \dots, I_{m-1}$ be chosen so that there exists an arc from I_k to I_{k+1} for all $k \in \mathcal{N}$.

Step 4. The map T admits periodic orbits of minimal period n for all $n > m$.
This follows from step 3 by applying Lemma 3.5 to the closed ammissible path $I_1 I_2 \dots I_{m-1} I_1 \dots I_1$ of length $n+1$.

Step 5. For each odd $k \in \mathcal{N}$ there exists an arc from I_{m-1} to I_k.
The statement is clearly true for $m = 3$. If $m > 3$ we show that the sets I_k are ordered in $[a, b]$ in a precise way. From step 3 we know that I_1 T-covers itself and I_2, and no other set. So $T(p_{\bar{h}}) = p_{\bar{h}+2}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}}$, or $T(p_{\bar{h}}) = p_{\bar{h}+1}$ and $T(p_{\bar{h}+1}) = p_{\bar{h}-1}$. In the first case $I_2 = [p_{\bar{h}+1}, p_{\bar{h}+2}]$, and since I_2 T-covers only I_3 we have $I_3 = [p_{\bar{h}-1}, p_{\bar{h}}]$. We can continue repeating the argument to conclude that $I_{m-1} = [p_{m-1}, p_m]$, and $T(p_{m-1}) = p_1$, $T(p_1) = p_m$ and $T(p_m) = p_{\bar{h}}$. Since I_k with k odd are of the form $[p_h, p_{h+1}]$ with $h < \bar{h}$, the thesis of the step follows.

Step 6. The map T admits periodic orbits of minimal period n for all even $n < m$.
This follows from step 5 by applying Lemma 3.5 to the closed admissible path of length $n+1$ from I_{m-1} to itself of the form $I_{m-1} I_j I_{j+1} \dots I_{m-1}$ where $j = m - n$ is odd. □

Theorem 3.8 (Sharkovsky)**.** *Let $T : [a, b] \to [a, b]$ be a continuous map and consider the following ordering on $\mathbb{N}$*

$$\begin{aligned} &1 \prec 2 \prec 4 \prec 8 \prec \dots \prec 2^n \prec 2^{n+1} \prec \dots 2^{n+1}5 \prec 2^{n+1}3 \prec \dots \\ &\dots \prec 2^n 5 \prec 2^n 3 \prec \dots \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \dots 7 \prec 5 \prec 3 \end{aligned} \tag{3.5}$$

If T admits a periodic orbit of minimal period m then it admits a periodic *orbit of minimal period n for all $n \prec m$ in the ordering* (3.5).

Proof. If m is odd, the thesis follows from Proposition 3.7.
If $m = 2 \cdot \tilde{m}$ with $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we can repeat the same argument of the proof of Proposition 3.7 up to step 2. This shows that $\bar{h} = \tilde{m}$ and, in the T-graph of the partition including the sets J_k with $k \in \mathcal{N}$, there exists an admissible path from the set $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ to all the sets J_k with $k \in \mathcal{N}$. This implies that T admits a fixed point. However there is no arc to $[p_{\tilde{m}}, p_{\tilde{m}+1}]$ from a different set, since otherwise by Lemma 3.5 we could find a periodic orbit of T with odd period. It follows that $T(p_j) \geq p_{\tilde{m}+1}$ for all $j \leq \tilde{m}$ and $T(p_j) \leq p_{\tilde{m}}$ for all $j \geq \tilde{m}+1$, so the points $p_1, \dots, p_{\tilde{m}}$ give a periodic orbit of period $\tilde{m}$ for T^2. We can then repeat the argument for T^2 and find periodic orbits of T^2 with period $\tilde{n}$ for all $\tilde{n} \prec \tilde{m}$ in the ordering (3.5). The thesis for T follows.
If $m = 2^r \cdot \tilde{m}$ with $r > 1$, $\tilde{m}$ odd and T admits no periodic orbits with odd period, then we do one step as in the previous case, and we are reduced to the case $m = 2^{r-1} \cdot \tilde{m}$. So we can repeat the argument and obtain the thesis. We remark that when $\tilde{m} = 1$, we only obtain periodic orbits with period powers of 2. □

3.3 Topological chaos

Definition 3.8. Let $T : X \to X$ be a continuous map on a metric space X. We say that T is *chaotic in the sense of Devaney* if there exists a compact forward invariant set $\Lambda \subset X$ such that:

(i) the set of periodic orbits is dense in Λ;

(ii) T is topologically transitive on Λ, that is for all open sets $U, V \subset X$ with non-empty intersection with Λ, there exists $n \in \mathbb{N}$ such that $T^n(U \cap \Lambda) \cap (V \cap \Lambda) \neq \emptyset$;

(iii) T has sensitive dependence on initial conditions on Λ, that is there exists $c > 0$ such that for all $x \in \Lambda$ and all $\varepsilon > 0$ one can find $y \in B_\varepsilon(x) \cap \Lambda$ for which there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) > c$.

Example 3.3. Show that the Symbolic dynamics of Example 1.8 is chaotic in the sense of Devaney.

Remark 3.9. Conditions (i) and (ii) in Definition 3.8 imply (iii) (see [Ru17, Thm 7.4]).

Definition 3.9. Let $T : X \to X$ be a continuous map on a compact metric space X. For $n \in \mathbb{N}$ and $\varepsilon > 0$, a set $S \subset X$ is called (n, ε)*-separated* if for all $x, y \in S$ there exists $k = 0, \dots, n$ such that $d(T^k(x), T^k(y)) > \varepsilon$. Then the quantity

$$h_{top}(T) := \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log \Big(\max \{ \#S \,:\, S \text{ is } (n, \varepsilon)\text{-separated} \} \Big)$$

is well-defined and is called *topological entropy of* T.

Proposition 3.10. *Let* (X, T) *and* $(\tilde{X}, \tilde{T})$ *be two discrete-time continuous dynamical systems on compact metric spaces, and assume that* $(\tilde{X}, \tilde{T})$ *is a topological factor of* (X, T). *Then* $h_{top}(T) \leq h_{top}(\tilde{T})$. *In particular, topological entropy is invariant under topological conjugacy.*

Example 3.4. Using Definition 3.9 and Proposition 3.10, show that: The Symbolic dynamics has positive topological entropy; The Tent map of Example 1.5 with $s = 2$, the Bernoulli map of Example 1.7, and the Logistic map of Example 1.6 with $\lambda = 4$ have topological entropy $\log 2$; The rotations of the circle of Example 1.4 have null topological entropy.

We now move to the case of maps of the interval. First, we give a simple criterion to compute the topological entropy in a special case.

Proposition 3.11. *Let $T : [a,b] \to [a,b]$ be a piecewise continuous monotone map with respect to a partition $\mathcal{J} = \{J_1, \dots, J_N\}$ of the compact interval $[a,b]$ into closed subintervals. Assume that $T(J_i) = [a,b]$ for all $i = 1, \dots, N$. Then*

$$h_{top}(T) = \lim_{k\to\infty} \frac{1}{k} \log\Big(\#Fix(T^k)\Big) = \log N\,.$$

We now introduce another notion of chaotic behaviour.

Definition 3.10. Let $T : X \to X$ be a continuous map on a compact interval $X = [a,b]$. We say that T has a *horseshoe* if there exists a closed sub-interval $J \subseteq X$ which T-covers itself 2-times.

Proposition 3.12. *Let $T : X \to X$ be a continuous map on a compact interval $X = [a,b]$. Then:*

(i) if T has a horseshoe then has periodic orbits with minimal period n for all $n \geq 1$;

(ii) if T has a periodic point with minimal odd period $m > 1$, then T^2 has a horseshoe.

Proof. (i) Let $J \subseteq [a,b]$ be the closed interval which covers itself 2-times, and let K_1 and K_2 be the open sub-intervals of J such that $K_1 \cap K_2 = \emptyset$ and $T(\overline{K_1}) = T(\overline{K_2}) = J$. We consider the T-graph of K_1, K_2, which is a full graph on the indices $\{1,2\}$.

Let $K_1 = (\alpha, \beta)$ and $K_2 = (\beta + \varepsilon, \gamma)$, there are two cases. If $\varepsilon > 0$ or $\varepsilon = 0$ and β is not a fixed point, we apply Lemma 3.5 to the admissible path $K_1K_2K_2K_1$ to find a periodic point of period 3 which is not fixed, so it has minimal period 3 and we can apply Sharkovsky Theorem 3.8. If $\varepsilon = 0$ and β is a fixed point, then it follows that there exists $\delta \in (\beta, \gamma)$ such that $T([\delta, \gamma]) = J$, so we can repeat the argument with $K_1 = (\alpha, \beta)$ and $K_3 = (\delta, \gamma)$.

(ii) Let m be the smallest odd number for which T has a periodic orbit of minimal period m, and let $\{p_1, \dots, p_{m-1}\}$ be the points of the periodic orbit in dynamical order, that is $T(p_i) = p_{i+1}$ for all $i = 1, \dots, m-2$, and $T(p_{m-1}) = p_1$. By Step 5 in the proof of Proposition 3.7, the point of the periodic orbit are ordered in $[a,b]$ as

$$a \leq p_{m-1} < p_{m-3} < \cdots < p_5 < p_3 < p_1 < p_2 < p_4 < \cdots < p_{m-4} < p_{m-2} \leq b$$

or specularly. In the first case, we find $T(p_1, p_2) = (p_3, p_2)$ so that there exists $\delta \in (p_1, p_2)$ such that $T(\delta) = p_1$, and hence $T^2(\delta) = p_2$. We now show that $J = [p_{m-1}, p_2]$ T^2-covers itself 2-times. Let $K_1 = (p_{m-1}, p_{m-3})$, then $T^2(p_{m-1}) = p_2$ and $T^2(p_{m-3}) = p_{m-1}$, hence $T^2(\overline{K_1}) = J$. If we also let $K_2 = (p_{m-3}, \delta)$, then as shown before again $T^2(\overline{K_2}) = J$. Since $K_1 \cap K_2 = \emptyset$, we are done. $\square$

Definition 3.11. Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. We say that T is *chaotic in the horseshoe sense* if there exists $n \in \mathbb{N}$ such that T^n has a horseshoe.

Theorem 3.13 ([Ru17], Thm 4.58 and Thm 7.3)**.** *Let $T : X \to X$ be a continuous map on a compact interval $X = [a, b]$. Then the following are equivalent:*

(i) T is chaotic in the sense of Devaney;

(ii) $h_{top}(T) > 0$;

(iii) T is chaotic in the horseshoe sense;

(iv) T has a periodic point with minimal period not a power of 2.

Example 3.5. The Tent map T_s of Example 1.5 is chaotic for all $s > 1$. If $s \geq \sqrt{2}$ one shows that T_s^2 has a horseshoe by using the interval $J_s = [\frac{1}{s+1}, \frac{s}{s+1}]$, since $\frac{1}{2} \in J_s$ and $T^2(\frac{1}{2}) \leq \frac{1}{s+1}$, whereas $T^2(\frac{1}{s+1}) = T^2(\frac{s}{s+1}) = \frac{s}{s+1}$. If $s \in (1, \sqrt{2})$, the result follows by observing that there exist intervals J_1 and J_2 on which T_s^2 is equal to T_{s^2} after rescaling.

Remark 3.14. For a $C^{1+\alpha}$ diffeomorphism of a manifold, positive topological entropy is equivalent to existence of a Smale horseshoe [Ka80].